

GREEN'S FORMULA WITH \mathbb{C}^* -ACTION AND CALDERO-KELLER'S FORMULA FOR CLUSTER ALGEBRAS

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ABSTRACT. It is known that Green's formula over finite fields gives rise to the comultiplications of Ringel-Hall algebras and quantum groups (see [Gre], see also [Lu]). In this paper, we prove a projective version of Green's formula in a geometric way. Then following the method of Hubery in [Hu2], we apply this formula to proving Caldero-Keller's multiplication formula for acyclic cluster algebras of arbitrary type.

1. INTRODUCTION

1.1. Green in [Gre] found a homological counting formula for hereditary abelian categories over finite fields. It leads to the comultiplication formula for Ringel-Hall algebras, and as a generalization of the result of Ringel in [Rin1], it gives a realization of the positive part of the quantized enveloping algebra for arbitrary type symmetrizable Kac-Moody algebra. In [DXX], we gave Green's formula over the complex numbers \mathbb{C} via Euler characteristic and applied it to realizing comultiplication of the universal enveloping algebra. However, one should notice that many nonzero terms in the original formula vanish when we consider it over the complex numbers \mathbb{C} . In the following, we show that the geometric correspondence in the proof of Green's formula admits a canonical \mathbb{C}^* -action. Then we obtain a new formula, which can be regarded as the projective version of Green's formula.

Our motivation comes from cluster algebras. Cluster algebras were introduced by Fomin and Zelevinsky [FZ]. In [BMRRT], the authors categorified a lot of cluster algebras by defining and studying the cluster categories related to clusters and seeds. Under the framework of cluster categories, Caldero and Keller realized the acyclic cluster algebras of simply-laced finite type by proving a cluster multiplication theorem [CK]. At the same time, Hubery researched on realizing acyclic cluster algebras (including non simply-laced case) via Ringel-Hall algebras for valued graphs over finite fields [Hu2]. He counted the corresponding Hall numbers and then deduced the Caldero-Keller multiplication when evaluating at $q = 1$ where q is the order of the finite field. It seems that his method only works for the case of tame hereditary algebras [Hu3], due to the difficulty of the existence of Hall polynomials. In this paper, we realize that the whole thing is independent of that over finite fields. By counting the Euler characteristics of the corresponding varieties and constructible sets with *pushforward* functors and geometric quotients, we show that the projective version of Green's theorem and the "higher order" associativity of Hall multiplication imply that Caldero-Keller's multiplication formula holds for acyclic cluster algebras of arbitrary type. We remark here that, for the elements in the dual semicanonical basis which are given by certain constructible functions

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on varieties of nilpotent modules over a preprojective algebra of arbitrary type, a similar multiplication formula has been obtained in [GLS].

1.2. The paper is organized as follows. In Section 2, we recall the general theory of algebraic geometry needed in this paper. This is followed in Section 3 by a short survey of Green's formula over finite fields without proof. In particular, we consider many variants of Green's formula under various group actions. These variants can be viewed as the counterparts over finite field of the projective version of Green's formula. We give the main result in Section 4. Two geometric versions of Green's formula are proved. As an application, in Section 5 we prove Caldero-Keller multiplication formula following Hubery's method [Hu2], and also we give an example using the Kronecker quiver.

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2. PRELIMINARIES

2.1. Let $Q = (Q_0, Q_1, s, t)$ be a quiver, where Q_0 , also denoted by I , and Q_1 are the sets of vertices and arrows, respectively, and $s, t : Q_1 \rightarrow Q_0$ are maps such that any arrow α starts at $s(\alpha)$ and terminates at $t(\alpha)$. For any dimension vector $\underline{d} = \sum_i a_i i \in NI$, we consider the affine space over \mathbb{C}

$$\mathbb{E}_{\underline{d}}(Q) = \bigoplus_{\alpha \in Q_1} \text{Hom}_{\mathbb{C}}(\mathbb{C}^{a_{s(\alpha)}}, \mathbb{C}^{a_{t(\alpha)}})$$

Any element $x = (x_{\alpha})_{\alpha \in Q_1}$ in $\mathbb{E}_{\underline{d}}(Q)$ defines a representation $M(x)$ with $\dim M(x) = \underline{d}$ in a natural way. For any $\alpha \in Q_1$, we denote the vector space at $s(\alpha)$ (resp. $t(\alpha)$) of the representation M by $M_{s(\alpha)}$ (resp. $M_{t(\alpha)}$) and the linear map from $M_{s(\alpha)}$ to $M_{t(\alpha)}$ by M_{α} . A relation in Q is a linear combination $\sum_{i=1}^r \lambda_i p_i$, where $\lambda_i \in \mathbb{C}$ and p_i are paths of length at least two with $s(p_i) = s(p_j)$ and $t(p_i) = t(p_j)$ for all $1 \leq i, j \leq r$. For any $x = (x_{\alpha})_{\alpha \in Q_1} \in \mathbb{E}_{\underline{d}}$ and any path $p = \alpha_m \cdots \alpha_2 \alpha_1$ in Q we set $x_p = x_{\alpha_m} \cdots x_{\alpha_2} x_{\alpha_1}$. Then x satisfies a relation $\sum_{i=1}^r \lambda_i p_i$ if $\sum_{i=1}^r \lambda_i x_{p_i} = 0$. If R is a set of relations in Q , then let $\mathbb{E}_{\underline{d}}(Q, R)$ be the closed subvariety of $\mathbb{E}_{\underline{d}}(Q)$ which consists of all elements satisfying all relations in R . Any element $x = (x_{\alpha})_{\alpha \in Q_1}$ in $\mathbb{E}_{\underline{d}}(Q, R)$ defines in a natural way a representation $M(x)$ of $A = \mathbb{C}Q/J$ with $\dim M(x) = \underline{d}$, where J is the admissible ideal generated by R . We consider the algebraic group

$$G_{\underline{d}}(Q) = \prod_{i \in I} GL(a_i, \mathbb{C}),$$

which acts on $\mathbb{E}_{\underline{d}}(Q)$ by $(x_{\alpha})^g = (g_{t(\alpha)} x_{\alpha} g_{s(\alpha)}^{-1})$ for $g \in G_{\underline{d}}$ and $(x_{\alpha}) \in \mathbb{E}_{\underline{d}}$. It naturally induces an action of $G_{\underline{d}}(Q)$ on $\mathbb{E}_{\underline{d}}(Q, R)$. The induced orbit space is denoted by $\mathbb{E}_{\underline{d}}(Q, R)/G_{\underline{d}}(Q)$. There is a natural bijection between the set $\mathcal{M}(A, \underline{d})$ of isomorphism classes of \mathbb{C} -representations of A with dimension vector \underline{d} and the set of orbits of $G_{\underline{d}}(Q)$ in $\mathbb{E}_{\underline{d}}(Q, R)$. So we may identify $\mathcal{M}(A, \underline{d})$ with $\mathbb{E}_{\underline{d}}(Q, R)/G_{\underline{d}}(Q)$.

The intersection of an open subset and a close subset in $\mathbb{E}_{\underline{d}}(Q, R)$ is called a locally closed subset. A subset in $\mathbb{E}_{\underline{d}}(Q, R)$ is called constructible if and only if it is a disjoint union of finitely many locally closed subsets. Obviously, an open set and a closed set are both constructible sets. A function f on $\mathbb{E}_{\underline{d}}(Q, R)$ is called constructible if $\mathbb{E}_{\underline{d}}(Q, R)$ can be divided into finitely many constructible sets such that f is constant on each such constructible set. Write $M(X)$ for the \mathbb{C} -vector space of constructible functions on some complex algebraic variety X .

Let \mathcal{O} be a constructible set as defined above. Let $1_{\mathcal{O}}$ be the characteristic function of \mathcal{O} , defined by $1_{\mathcal{O}}(x) = 1$, for any $x \in \mathcal{O}$ and $1_{\mathcal{O}}(x) = 0$, for any $x \notin \mathcal{O}$.

It is clear that $1_{\mathcal{O}}$ is the simplest constructible function and any constructible function is a linear combination of characteristic functions. For any constructible subset \mathcal{O} in $\mathbb{E}_{\underline{d}}(Q, R)$, we call \mathcal{O} $G_{\underline{d}}$ -invariant if $G_{\underline{d}} \cdot \mathcal{O} = \mathcal{O}$.

In the following, we will always assume constructible sets and functions to be $G_{\underline{d}}$ -invariant unless particular stated.

2.2. Let χ denote Euler characteristic in compactly-supported cohomology. Let X be an algebraic variety and \mathcal{O} a constructible subset which is the disjoint union of finitely many locally closed subsets X_i for $i = 1, \dots, m$. Define $\chi(\mathcal{O}) = \sum_{i=1}^m \chi(X_i)$. Note that it is well-defined. We will use the following properties:

Proposition 2.1 ([Rie] and [Joy]). *Let X, Y be algebraic varieties over \mathbb{C} . Then*

- (1) *If the algebraic variety X is the disjoint union of finitely many constructible sets X_1, \dots, X_r , then*

$$\chi(X) = \sum_{i=1}^r \chi(X_i).$$

- (2) *If $\varphi : X \rightarrow Y$ is a morphism with the property that all fibers have the same Euler characteristic χ , then $\chi(X) = \chi \cdot \chi(Y)$. In particular, if φ is a locally trivial fibration in the analytic topology with fibre F , then $\chi(X) = \chi(F) \cdot \chi(Y)$.*
- (3) *$\chi(\mathbb{C}^n) = 1$ and $\chi(\mathbb{P}^n) = n + 1$ for all $n \geq 0$.*

We recall the definition *pushforward* functor from the category of algebraic varieties over \mathbb{C} to the category of \mathbb{Q} -vector spaces (see [Mac] and [Joy]).

Let $\phi : X \rightarrow Y$ be a morphism of varieties. For $f \in M(X)$ and $y \in Y$, define

$$\phi_*(f)(y) = \sum_{c \in \mathbb{Q}} c \chi(f^{-1}(c) \cap \phi^{-1}(y))$$

Theorem 2.2 ([Di],[Joy]). *Let X, Y and Z be algebraic varieties over \mathbb{C} , $\phi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ be morphisms of varieties, and $f \in M(X)$. Then $\phi_*(f)$ is constructible, $\phi_* : M(X) \rightarrow M(Y)$ is a \mathbb{Q} -linear map and $(\psi \circ \phi)_* = (\psi)_* \circ (\phi)_*$ as \mathbb{Q} -linear maps from $M(X)$ to $M(Z)$.*

In order to deal with orbit spaces, we need to consider geometric quotients.

Definition 2.3. *Let G be an algebraic group acting on a variety X and $\phi : X \rightarrow Y$ be a G -invariant morphism, i.e. a morphism constant on orbits. The pair (Y, ϕ) is called a geometric quotient if ϕ is open and for any open subset U of Y , the associated comorphism identifies the ring $\mathcal{O}_Y(U)$ of regular functions on U with the ring $\mathcal{O}_X(\phi^{-1}(U))^G$ of G -invariant regular functions on $\phi^{-1}(U)$.*

The following result due to Rosenlicht [Ro] is essential to us.

Lemma 2.4. *Let X be a G -variety, then there exists an open and dense G -stable subset which has a geometric G -quotient.*

By this Lemma, we can construct a finite stratification over X . Let U_1 be an open and dense G -stable subset of X as in Lemma 2.4. Then $\dim_{\mathbb{C}}(X - U_1) < \dim_{\mathbb{C}} X$. We can use the above lemma again, there exists a dense open G -stable subset U_2 of $X - U_1$ which has a geometric G -quotient. Inductively, we get a finite stratification $X = \cup_{i=1}^l U_i$ where U_i is a G -invariant locally closed subset and has a geometric quotient, $l \leq \dim_{\mathbb{C}} X$. We denote by ϕ_{U_i} the geometric quotient map on U_i . Define the quasi Euler-Poincaré characteristic of X/G by $\chi(X/G) := \sum_i \chi(\phi_{U_i}(U_i))$. If $\{U'_i\}$ is another choice in the definition of $\chi(X/G)$, then $\chi(\phi_{U_i}(U_i)) = \sum_j \chi(\phi_{U_i \cap U'_j}(U_i \cap U'_j))$ and $\chi(\phi_{U'_j}(U'_j)) = \sum_i \chi(\phi_{U_i \cap U'_j}(U_i \cap U'_j))$.

Thus $\sum_i \chi(\phi_{U_i}(U_i)) = \sum_i \chi(\phi_{U'_i}(U'_i))$ and $\chi(X/G)$ is well-defined (see [XXZ]). Similarly, $\chi(\mathcal{O}/G) := \sum_i \chi(\phi_{U_i}(\mathcal{O} \cap U_i))$ is well-defined for any G -invariant constructible subset \mathcal{O} of X .

2.3. We also introduce the following notation. Let f be a constructible function over a variety X , it is natural to define

$$(1) \quad \int_{x \in X} f(x) := \sum_{m \in \mathbb{C}} m \chi(f^{-1}(m))$$

Comparing with Proposition 2.1, we also have the following (see [XXZ]).

Proposition 2.5. *Let X, Y be algebraic varieties over \mathbb{C} under the actions of the algebraic groups G and H , respectively. Then*

- (1) *If the algebraic variety X is the disjoint union of finitely many G -invariant constructible sets X_1, \dots, X_r , then*

$$\chi(X/G) = \sum_{i=1}^r \chi(X_i/G)$$

- (2) *If a morphism $\varphi : X \rightarrow Y$ induces a quotient map $\phi : X/G \rightarrow Y/H$ whose fibers all have the same Euler characteristic χ , then $\chi(X/G) = \chi \cdot \chi(Y/H)$.*

Moreover, if there exists an action of an algebraic group G on X as in Definition 2.3, and f is a G -invariant constructible function over X , we define

$$(2) \quad \int_{x \in X/G} f(x) := \sum_{m \in \mathbb{C}} m \chi(f^{-1}(m)/G)$$

In particular, we frequently use the following corollary.

Corollary 2.6. *Let X, Y be algebraic varieties over \mathbb{C} under the actions of an algebraic group G . These actions naturally induce an action of G on $X \times Y$. Then*

$$\chi(X \times_G Y) = \int_{y \in Y/G} \chi(X/G_y)$$

where G_y is the stabilizer in G of $y \in Y$ and $X \times_G Y$ is the orbit space of $X \times Y$ under the action of G .

3. GREEN'S FORMULA OVER FINITE FIELDS

3.1. In this section, we recall Green's formula over finite fields ([Gre], [Rin2]). Let k be a finite field and Λ a hereditary finitary k -algebra, i.e., $\text{Ext}^1(M, N)$ is a finite set and $\text{Ext}^2(M, N) = 0$ for any Λ -modules M, N . Let \mathcal{P} be the set of isomorphism classes of finite Λ -modules. Let $\mathcal{H}(\Lambda)$ be the Ringel-Hall algebra associated to $\text{mod } \Lambda$. Green introduced on \mathcal{H} a comultiplication so that \mathcal{H} becomes a bialgebra up to a twist on $\mathcal{H} \otimes \mathcal{H}$. His proof of the compatibility between the multiplication and the comultiplication completely depends on the following Green's formula.

Given $\alpha \in \mathcal{P}$, let V_α be a representative in α , and $a_\alpha = |\text{Aut}_\Lambda V_\alpha|$. Given ξ, η and λ in \mathcal{P} , let $g_{\xi\eta}^\lambda$ be the number of submodules Y of V_λ such that Y and V_λ/Y belong to η and ξ , respectively.

Theorem 3.1. *Let k be a finite field and Λ a hereditary finitary k -algebra. Let $\xi, \eta, \xi', \eta' \in \mathcal{P}$. Then*

$$a_\xi a_\eta a_{\xi'} a_{\eta'} \sum_{\lambda} g_{\xi\eta}^\lambda g_{\xi'\eta'}^\lambda a_\lambda^{-1} = \sum_{\alpha, \beta, \gamma, \delta} \frac{|\text{Ext}^1(V_\gamma, V_\beta)|}{|\text{Hom}(V_\gamma, V_\beta)|} g_{\gamma\alpha}^\xi g_{\gamma\delta}^{\xi'} g_{\delta\beta}^\eta g_{\alpha\beta}^{\eta'} a_\alpha a_\beta a_\delta a_\gamma$$

Suppose $X \in \xi, Y \in \eta, M \in \xi', N \in \eta'$ and $A \in \gamma, C \in \alpha, B \in \delta, D \in \beta, E \in \lambda$. Set $h_\lambda^{\xi\eta} := |\text{Ext}^1(X, Y)_E|$, where $\text{Ext}^1(X, Y)_E$ is the subset of $\text{Ext}^1(X, Y)$ consisting of elements ω such that the middle term of an exact sequence represented by ω is isomorphic to E . Then the above formula can be rewritten as ([DXX], [Hu2])

$$\sum_{\lambda} g_{\xi\eta}^{\lambda} h_{\lambda}^{\xi'\eta'} = \sum_{\alpha, \beta, \gamma, \delta} \frac{|\text{Ext}^1(A, D)| |\text{Hom}(M, N)|}{|\text{Hom}(A, D)| |\text{Hom}(A, C)| |\text{Hom}(B, D)|} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} h_{\xi}^{\gamma\alpha} h_{\eta}^{\delta\beta}$$

3.2. For fixed kQ -modules X, Y, M, N with $\dim X + \dim Y = \dim M + \dim N$, we fix a Q_0 -graded k -space E such that $\dim E = \dim X + \dim Y$. Let (E, m) be the kQ -module structure on E given by an algebraic morphism $m : \Lambda \rightarrow \text{End}_k E$. Let $Q(E, m)$ be the set of (a, b, a', b') such that the row and the column of the following diagram are exact:

$$(3) \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & Y & & & \\ & & & \downarrow a' & & & \\ 0 & \longrightarrow & N & \xrightarrow{a} & (E, m) & \xrightarrow{b} & M \longrightarrow 0 \\ & & & & \downarrow b' & & \\ & & & & X & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Let

$$Q(X, Y, M, N) = \bigcup_{m: \Lambda \rightarrow \text{End}_k E} Q(E, m)$$

It is clear that

$$|Q(E, m)| = g_{\xi\eta}^{\lambda} g_{\xi'\eta'}^{\lambda} a_{\xi} a_{\eta} a_{\xi'} a_{\eta'}$$

where $\lambda \in \mathcal{P}$ is such that $(E, m) \in \lambda$, or simply write $m \in \lambda$.

$$|Q(X, Y, M, N)| = \sum_{\lambda} \frac{|\text{Aut}_k E|}{a_{\lambda}} g_{\xi\eta}^{\lambda} g_{\xi'\eta'}^{\lambda} a_{\xi} a_{\eta} a_{\xi'} a_{\eta'}$$

There is an action of $\text{Aut}_{\Lambda}(E, m)$ on $Q(E, m)$ given by

$$g \cdot (a, b, a', b') = (ga, bg^{-1}, ga', b'g^{-1})$$

This induces an orbit space of $Q(E, m)$, denoted by $Q(E, m)^*$. The orbit of (a, b, a', b') in $Q(E, m)^*$ is denoted by $(a, b, a', b')^*$. We have

$$|Q(X, Y, M, N)| = |\text{Aut}_k E| \sum_{\lambda \in \mathcal{P}} \sum_{(a, b, a', b')^* \in Q(E, m)^*, m \in \lambda} \frac{1}{|\text{Hom}(\text{Coker } b'a, \text{Ker } ba')|}$$

Furthermore, there is an action of the group $\text{Aut } X \times \text{Aut } Y$ on $Q(E, m)^*$ given by

$$(g_1, g_2) \cdot (a, b, a', b')^* = (a, b, a'g_2^{-1}, g_1b')^*$$

for $(g_1, g_2) \in \text{Aut } X \times \text{Aut } Y$ and $(a, b, a', b')^* \in Q(E, m)^*$. The stabilizer $G((a, b, a', b')^*)$ of $(a, b, a', b')^*$ is

$$\{(g_1, g_2) \in \text{Aut } X \times \text{Aut } Y \mid ga' = a'g_2, b'g = g_1b' \text{ for some } g \in 1 + a\text{Hom}(M, N)b\}$$

The orbit space is denoted by $Q(E, m)^\wedge$ and the orbit of $(a, b, a', b')^*$ is denoted by $(a, b, a', b')^\wedge$. We have

$$\frac{1}{a_X a_Y} |Q(E, m)^*| = \sum_{(a, b, a', b')^\wedge \in Q(E, m)^\wedge} \frac{1}{|G((a, b, a', b')^*)|}$$

3.3. Let $\mathcal{D}(X, Y, M, N)^*$ be the set of $(B, D, e_1, e_2, e_3, e_4)$ such that the following diagram has exact rows and exact columns:

$$(4) \quad \begin{array}{ccccccc} & & 0 & & & 0 & \\ & & \downarrow & & & \downarrow & \\ 0 & \longrightarrow & D & \xrightarrow{e_1} & Y & \xrightarrow{e_2} & B \longrightarrow 0 \\ & & \downarrow u' & & & \downarrow x & \\ & & N & & & M & \\ & & \downarrow v' & & & \downarrow y & \\ 0 & \longrightarrow & C & \xrightarrow{e_3} & X & \xrightarrow{e_4} & A \longrightarrow 0 \\ & & \downarrow & & & \downarrow & \\ & & 0 & & & 0 & \end{array}$$

where B, D are submodules of M, N , respectively and $A = M/B, C = N/D$. The maps u', v' and x, y are naturally induced. We have

$$|\mathcal{D}(X, Y, M, N)^*| = \sum_{\alpha, \beta, \gamma, \delta} g_{\gamma\alpha}^\xi g_{\gamma\delta}^{\xi'} g_{\delta\beta}^\eta g_{\alpha\beta}^{\eta'} a_\alpha a_\beta a_\delta a_\gamma$$

There is an action of the group $\text{Aut}_\Lambda X \times \text{Aut}_\Lambda Y$ on $\mathcal{D}(X, Y, M, N)^*$ given by

$$(g_1, g_2) \cdot (B, D, e_1, e_2, e_3, e_4) = (B, D, g_2 e_1, e_2 g_2^{-1}, g_1 e_3, e_4 g_1^{-1})$$

for $(g_1, g_2) \in \text{Aut}_\Lambda X \times \text{Aut}_\Lambda Y$. The orbit space is denoted by $\mathcal{D}(X, Y, M, N)^\wedge$. We have

$$|\mathcal{D}(X, Y, M, N)^\wedge| = \frac{1}{a_X a_Y} \sum_{\alpha, \beta, \gamma, \delta} |\text{Hom}(A, C)| |\text{Hom}(B, D)| g_{\gamma\alpha}^\xi g_{\gamma\delta}^{\xi'} g_{\delta\beta}^\eta g_{\alpha\beta}^{\eta'} a_\alpha a_\beta a_\delta a_\gamma$$

Fix a square as above, let $T = X \times_A M = \{(x \oplus m) \in X \oplus M \mid e_4(x) = y(m)\}$ and $S = Y \sqcup_D N = Y \oplus N / \{e_1(d) \oplus u'(d) \mid d \in D\}$. There is a unique map $f : S \rightarrow T$ (see [Rin2]) such that the natural long sequence

$$(5) \quad 0 \rightarrow D \rightarrow S \xrightarrow{f} T \rightarrow A \rightarrow 0$$

is exact.

Let (c, d) be a pair of maps such that c is surjective, d is injective and $cd = f$. The number of such pairs can be computed as follows. We have the following commutative diagram:

$$(6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & S & \xrightarrow{d} & (E, m) & \xrightarrow{d_1} & A \longrightarrow 0 \\ & & \downarrow & & \downarrow c & & \parallel \\ \varepsilon_0 : & 0 & \longrightarrow & \text{Im } f & \longrightarrow & T & \longrightarrow A \longrightarrow 0 \end{array}$$

The exact sequence

$$0 \longrightarrow D \longrightarrow S \longrightarrow \text{Im} f \longrightarrow 0$$

induces the following long exact sequence:

$$(7) \quad 0 \longrightarrow \text{Hom}(A, D) \longrightarrow \text{Hom}(A, S) \longrightarrow \text{Hom}(A, \text{Im} f) \longrightarrow \\ \longrightarrow \text{Ext}^1(A, D) \longrightarrow \text{Ext}^1(A, S) \xrightarrow{\phi} \text{Ext}^1(A, \text{Im} f) \longrightarrow 0$$

We set $\varepsilon_0 \in \text{Ext}^1(A, \text{Im} f)$ corresponding to the canonical exact sequence

$$0 \longrightarrow \text{Im} f \longrightarrow T \longrightarrow A \longrightarrow 0$$

and denote $\phi^{-1}(\varepsilon_0) \cap \text{Ext}^1(A, S)_{(E, m)}$ by $\phi_m^{-1}(\varepsilon_0)$. Let $\mathcal{F}(f; m)$ be the set of (c, d) induced by diagram (6) with center term (E, m) . Let

$$\mathcal{F}(f) = \bigcup_{m: \Lambda \rightarrow \text{End}_k E} \mathcal{F}(f; m)$$

Then

$$|\mathcal{F}(f; m)| = |\phi_m^{-1}(\varepsilon_0)| \frac{|\text{Aut}_\Lambda(E, m)|}{|\text{Hom}(A, S)|} |\text{Hom}(A, \text{Im} f)|, \\ |\mathcal{F}(f)| = |\text{Aut}_k(E)| \frac{|\text{Ext}^1(A, D)|}{|\text{Hom}(A, D)|}.$$

Let $\mathcal{O}(E, m)$ be the set of $(B, D, e_1, e_2, e_3, e_4, c, d)$ such that the following diagram is commutative and has exact rows and columns:

$$(8) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D & \xrightarrow{e_1} & Y & \xrightarrow{e_2} & B \longrightarrow 0 \\ & & \downarrow u' & \nearrow u_Y & \downarrow & & \downarrow x \\ & & S & \xleftarrow{u_N} & (E, m) & \xrightarrow{c} & M \\ & & \downarrow u' & \nearrow d & \downarrow & & \downarrow y \\ & & N & \xrightarrow{v'} & T & \xrightarrow{q_M} & A \\ & & \downarrow v' & \nearrow q_X & \downarrow & & \downarrow \\ 0 & \longrightarrow & C & \xrightarrow{e_3} & X & \xrightarrow{e_4} & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where the maps q_X, u_Y and q_M, u_N are naturally induced. In fact, the long exact sequence (5) has the following explicit form:

$$(9) \quad 0 \longrightarrow D \xrightarrow{u_Y e_1} S \xrightarrow{cd} T \xrightarrow{e_4 q_X} A \longrightarrow 0$$

$$|\mathcal{O}(E, m)| = \sum_{\alpha, \beta, \gamma, \delta} |\phi_m^{-1}(\varepsilon_0)| \frac{|\text{Aut}_\Lambda(E, m)|}{|\text{Hom}(A, S)|} |\text{Hom}(A, \text{Im} f)| g_{\gamma\alpha}^\xi g_{\gamma\delta}^{\xi'} g_{\delta\beta}^\eta g_{\alpha\beta}^{\eta'} a_\alpha a_\beta a_\delta a_\gamma$$

Let $\mathcal{O}(X, Y, M, N) = \bigcup_{m: \Lambda \rightarrow \text{End}_k E} \mathcal{O}(E, m)$

$$|\mathcal{O}(X, Y, M, N)| = |\text{Aut}_k(E)| \sum_{\alpha, \beta, \gamma, \delta} \frac{|\text{Ext}^1(A, D)|}{|\text{Hom}(A, D)|} g_{\gamma\alpha}^\xi g_{\gamma\delta}^{\xi'} g_{\delta\beta}^\eta g_{\alpha\beta}^{\eta'} a_\alpha a_\beta a_\delta a_\gamma$$

The group $\text{Aut}_\Lambda(E, m)$ naturally acts on $\mathcal{O}(E, m)$ and $\mathcal{O}(X, Y, M, N)$ as follows:

$$g.(B, D, e_1, e_2, e_3, e_4, c, d) = (B, D, e_1, e_2, e_3, e_4, cg^{-1}, gd)$$

We denote the orbit space by $\mathcal{O}(E, m)^*$ and $\mathcal{O}(X, Y, M, N)^*$. The orbit of $(B, D, e_1, e_2, e_3, e_4, c, d)$ is denoted by $(B, D, e_1, e_2, e_3, e_4, c, d)^*$. Then

$$|\mathcal{O}(X, Y, M, N)^*| = \sum_{\alpha, \beta, \gamma, \delta} |\text{Ext}^1(A, D)| g_{\gamma\alpha}^\xi g_{\gamma\delta}^{\xi'} g_{\delta\beta}^\eta g_{\alpha\beta}^{\eta'} a_\alpha a_\beta a_\delta a_\gamma$$

Similar to that on $\mathcal{D}(X, Y, M, N)^*$, there is an action of $\text{Aut } X \times \text{Aut } Y$ on $\mathcal{O}(X, Y, M, N)^*$ given by

$$(g_1, g_2).(B, D, e_1, e_2, e_3, e_4, c, d)^* = (B, D, g_2 e_1, e_2 g_2^{-1}, g_1 e_3, e_4 g_1^{-1}, c', d')^*$$

Let us determine the relation between (c', d') and (c, d) .

It is clear that there are isomorphisms:

$$a_1 : S \rightarrow S' \quad \text{and} \quad a_2 : T \rightarrow T'$$

induced by isomorphisms:

$$\begin{pmatrix} g_2 & 0 \\ 0 & id \end{pmatrix} : Y \oplus N \rightarrow Y \oplus N \quad \text{and} \quad \begin{pmatrix} g_1 & 0 \\ 0 & id \end{pmatrix} : X \oplus M \rightarrow X \oplus M$$

Hence, $c' = a_2 c, d' = d a_1^{-1}$.

The stabilizer of $(B, D, e_1, e_2, e_3, e_4, c, d)^*$ is denoted by $G((B, D, e_1, e_2, e_3, e_4, c, d)^*)$, which is

$$\{(g_1, g_2) \in \text{Aut } X \times \text{Aut } Y \mid g_1 \in e_3 \text{Hom}(A, C) e_4, g_2 \in e_1 \text{Hom}(B, D) e_2, \\ cg^{-1} = c', gd = d' \text{ for some } g \in \text{Aut}(E, m)\}$$

The orbit space is denoted by $\mathcal{O}(X, Y, M, N)^\wedge$ and the orbit is denoted by $(B, D, e_1, e_2, e_3, e_4, c, d)^\wedge$. Then

$$\frac{1}{a_X a_Y} |\mathcal{O}(X, Y, M, N)^*| = \sum_{(B, D, e_1, e_2, e_3, e_4, c, d)^\wedge \in \mathcal{O}(X, Y, M, N)^\wedge} \frac{1}{|G((B, D, e_1, e_2, e_3, e_4, c, d)^*)|}$$

3.4. There is a bijection $\Omega : Q(E, m) \rightarrow \mathcal{O}(E, m)$ which induces Green's formula. In the same way, we also have the following proposition

Proposition 3.2. *There exist bijections $\Omega^* : Q(E, m)^* \rightarrow \mathcal{O}(E, m)^*$ and $\Omega^\wedge : Q(E, m)^\wedge \rightarrow \mathcal{O}(E, m)^\wedge$.*

Proof. For any $(a, b, a', b') \in Q(E, m)$,

$$\Omega(a, b, a', b') = (\text{Ker } b'a, \text{Im } ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d)$$

where c, d are defined by

$$du_N = a, du_Y = a', q_M c = b, q_X c = b'$$

Hence,

$$\begin{aligned} \Omega(g.(a, b, a', b')) &= (ga, bg^{-1}, ga', b'g^{-1}) \\ &= (\text{Ker } b'a, \text{Im } ba', a'^{-1}a, ba', b'a, bb'^{-1}, cg^{-1}, gd) \\ &= g.(\text{Ker } b'a, \text{Im } ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d) \end{aligned}$$

i.e.

$$\Omega^*((a, b, a', b')^*) = ((\text{Ker } b'a, \text{Im } ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d))^*$$

for $g \in \text{Aut}(E, m)$. Similarly,

$$\begin{aligned} \Omega^*((g_1, g_2) \cdot (a, b, a', b')^*) &= (a, b, a'g_2^{-1}, g_1b')^* \\ &= (\text{Ker } g_1b'a, \text{Im } ba'g_2^{-1}, g_2a'^{-1}a, ba'g_2^{-1}, g_1b'a, bb'^{-1}g_1^{-1}, c', d')^* \\ &= (g_1, g_2) \cdot (\text{Ker } b'a, \text{Im } ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d)^* \end{aligned}$$

for $(g_1, g_2) \in \text{Aut } X \times \text{Aut } Y$. Hence

$$\Omega^\wedge((a, b, a', b')^\wedge) = ((\text{Ker } b'a, \text{Im } ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d)^\wedge)^\wedge$$

□

In particular, if $(a, b, a', b')^*$ corresponds to $(B, D, e_1, e_2, e_3, e_4, c, d)^*$, then

$$G((a, b, a', b')^*) = G((B, D, e_1, e_2, e_3, e_4, c, d)^*)$$

We also give the following variant of Green's formula, which is suggestive for the projective Green's formula over the complex numbers in the next section.

$$\begin{aligned} \sum_{\lambda; \lambda \neq \xi' \oplus \eta'} \frac{1}{q-1} h_\lambda^{\xi' \eta'} g_{\xi \eta}^\lambda &= \\ &\sum_{\alpha, \beta, \gamma, \delta; \alpha \oplus \gamma \neq \xi \text{ or } \beta \oplus \delta \neq \eta} \frac{|\text{Ext}^1(A, D)| |\text{Hom}(M, N)|}{|\text{Hom}(A, D)| |\text{Hom}(A, C)| |\text{Hom}(B, D)|} \frac{1}{q-1} h_\xi^{\gamma \alpha} h_\eta^{\delta \beta} g_{\gamma \delta}^{\xi'} g_{\alpha \beta}^{\eta'} \\ &+ \sum_{\alpha, \beta, \gamma, \delta; \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} \frac{1}{q-1} \left(\frac{|\text{Ext}^1(A, D)| |\text{Hom}(M, N)|}{|\text{Hom}(A, D)| |\text{Hom}(A, C)| |\text{Hom}(B, D)|} - 1 \right) g_{\gamma \delta}^{\xi'} g_{\alpha \beta}^{\eta'} \\ &+ \frac{1}{q-1} \left(\sum_{\alpha, \beta, \gamma, \delta; \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} g_{\gamma \delta}^{\xi'} g_{\alpha \beta}^{\eta'} - g_{\xi \eta}^{\xi' \oplus \eta'} \right) \end{aligned}$$

4. GREEN'S FORMULA OVER THE COMPLEX NUMBERS

4.1. From now on, we consider $A = \mathbb{C}Q$, where \mathbb{C} is the field of complex numbers. Let $\mathcal{O}_1, \mathcal{O}_2$ be G -invariant constructible subsets in $\mathbb{E}_{\underline{d}_1}(Q), \mathbb{E}_{\underline{d}_2}(Q)$, respectively, and let $\underline{d} = \underline{d}_1 + \underline{d}_2$. Define

$$\mathcal{V}(\mathcal{O}_1, \mathcal{O}_2; L) = \{0 = X_0 \subseteq X_1 \subseteq X_2 = L \mid X_i \in \text{mod } A, X_1 \in \mathcal{O}_2, \text{ and } L/X_1 \in \mathcal{O}_1\}.$$

where $L \in \mathbb{E}_{\underline{d}}(Q)$. In particular, when $\mathcal{O}_1, \mathcal{O}_2$ are the orbits of A -modules X, Y respectively, we write $\mathcal{V}(X, Y; L)$ instead of $\mathcal{V}(\mathcal{O}_1, \mathcal{O}_2; L)$.

Let α be the image of X in $\mathbb{E}_{\underline{d}_\alpha}(Q)/G_{\underline{d}_\alpha}$. We write $X \in \alpha$, sometimes we also use the notation \overline{X} to denote the image of X and the notation V_α to denote a representative of α . Instead of \underline{d}_α , we use $\underline{\alpha}$ to denote the dimension vector of α . Put

$$g_{\alpha \beta}^\lambda = \chi(\mathcal{V}(X, Y; L))$$

for $X \in \alpha, Y \in \beta$ and $L \in \lambda$. Both are well-defined and independent of the choice of objects in the orbits.

Definition 4.1. [Rie] For any $L \in \text{mod } A$, let $L = \bigoplus_{i=1}^r L_i$ be the decomposition into indecomposables, then an action of \mathbb{C}^* on L is defined by

$$t \cdot (v_1, \dots, v_r) = (tv_1, \dots, t^r v_r)$$

for $t \in \mathbb{C}^*$ and $v_i \in L_i$ for $i = 1, \dots, r$.

It induces an action of \mathbb{C}^* on $\mathcal{V}(X, Y; L)$ for any A -modules X, Y and L . Let $(X_1 \subseteq L) \in \mathcal{V}(X, Y; L)$ and $t \cdot X_1$ be the action of \mathbb{C}^* on X_1 as above under the decomposition of L , then there is a natural isomorphism between A -modules $t_{X_1} : X_1 \simeq t \cdot X_1$. Define $t \cdot (X_1 \subseteq L) = (t \cdot X_1 \subseteq L)$.

Let $D(X, Y)$ be the vector space over \mathbb{C} of all tuples $d = (d(\alpha))_{\alpha \in Q_1}$ such that each linear map $d(\alpha)$ belongs to $\text{Hom}_{\mathbb{C}}(X_{s(\alpha)}, Y_{t(\alpha)})$. Define $\pi : D(X, Y) \rightarrow \text{Ext}^1(X, Y)$ by sending d to the short exact sequence

$$\varepsilon : 0 \longrightarrow Y \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} L(d) \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} X \longrightarrow 0$$

where $L(d)$ is the direct sum of X and Y as a vector space and for any $\alpha \in Q_1$,

$$L(d)_\alpha = \begin{pmatrix} Y_\alpha & d(\alpha) \\ 0 & X_\alpha \end{pmatrix}$$

Fix a vector space decomposition $D(X, Y) = \text{Ker}\pi \oplus E(X, Y)$, then we can identify $\text{Ext}^1(X, Y)$ with $E(X, Y)$ (see [Rie], [DXX] or [GLS]). There is a natural \mathbb{C}^* -action on $E(X, Y)$ given by $t.d = (td(\alpha))$ for any $t \in \mathbb{C}^*$. This induces an action of \mathbb{C}^* on $\text{Ext}^1(X, Y)$. By the isomorphism of $\mathbb{C}Q$ -modules between $L(d)$ and $L(t.d)$, $t.\varepsilon$ is the following short exact sequence:

$$0 \longrightarrow Y \xrightarrow{\begin{pmatrix} t \\ 0 \end{pmatrix}} L(d) \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} X \longrightarrow 0$$

for any $t \in \mathbb{C}^*$. Let $\text{Ext}^1(X, Y)_L$ be the subset of $\text{Ext}^1(X, Y)$ of the equivalence classes of short exact sequences whose middle term is isomorphic to L . Then $\text{Ext}^1(X, Y)_L$ can be viewed as a constructible subset of $\text{Ext}^1(X, Y)$ under the identification between $\text{Ext}^1(X, Y)$ and $E(X, Y)$. Put

$$h_\lambda^{\alpha\beta} = \chi(\text{Ext}_A^1(X, Y)_L)$$

for $X \in \alpha, Y \in \beta$ and $L \in \lambda$. The following is known, for example, see [DXX].

Lemma 4.2. *For $A, B, X \in \text{mod}\Lambda$, $\chi(\text{Ext}_\Lambda^1(A, B)_X) = 0$ unless $X \simeq A \oplus B$.*

We remark that both $\mathcal{V}(X, Y; L)$ and $\text{Ext}^1(X, Y)_L$ can be viewed as the orbit spaces of

$W(X, Y; L) := \{(f, g) \mid 0 \longrightarrow Y \xrightarrow{f} L \xrightarrow{g} X \longrightarrow 0 \text{ is an exact sequence}\}$
under the actions of $G_\alpha \times G_\beta$ and G_λ respectively, for $X \in \alpha, Y \in \beta$ and $L \in \lambda$.

4.2. For fixed ξ, η, ξ', η' , consider the following canonical embedding:

$$(10) \quad \bigcup_{\alpha, \beta, \gamma, \delta; \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} \mathcal{V}(V_\alpha, V_\beta; V_{\eta'}) \times \mathcal{V}(V_\gamma, V_\delta; V_{\xi'}) \xrightarrow{i} \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})$$

sending $(V_{\eta'}^1 \subseteq V_{\eta'}, V_{\xi'}^1 \subseteq V_{\xi'})$ to $(V_{\xi'}^1 \oplus V_{\eta'}^1 \subseteq V_{\xi'} \oplus V_{\eta'})$ in a natural way. We set

$$\overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) := \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \setminus \text{Im}i$$

i.e.

$$(11) \quad \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) = \overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \bigcup \mathcal{V}_1$$

where $\mathcal{V}_1 = \text{Im}i$. Define

$$\mathcal{V}_1(\delta, \beta) := \text{Im}(\mathcal{V}(V_\alpha, V_\beta; V_{\eta'}) \times \mathcal{V}(V_\gamma, V_\delta; V_{\xi'}))$$

Consider the \mathbb{C} -space $M_G(A) = \bigoplus_{\underline{d} \in \mathbb{N}^n} M_{G_{\underline{d}}}(Q)$ where $M_{G_{\underline{d}}}(Q)$ is the \mathbb{C} -space of $G_{\underline{d}}$ -invariant constructible function on $\mathbb{E}_{\underline{d}}(Q)$. Define the convolution multiplication on $M_G(A)$ by

$$f \bullet g(L) = \sum_{c,d \in \mathbb{C}} \chi(\mathcal{V}(f^{-1}(c), g^{-1}(d); L)) cd.$$

for any $f \in M_{G_{\underline{d}}}(Q)$, $g \in M_{G_{\underline{d}'}}(Q)$ and $L \in \mathbb{E}_{\underline{d}+\underline{d}'}$.

As usual for an algebraic variety V and a constructible function f on V , using the notation (1) in Section 2, we have

$$f \bullet g(L) = \int_{\mathcal{V}(\text{supp}(f), \text{supp}(g); L)} f(x')g(x'').$$

The following is well-known (see [Lu],[Rie]), see a proof in [DXX].

Proposition 4.3. *The space $M_G(\Lambda)$ under the convolution multiplication \bullet is an associative \mathbb{C} -algebra with unit element.*

The above proposition implies the following identity

Theorem 4.4. *For fixed A -modules X, Y, Z and M with dimension vectors $\underline{d}_X, \underline{d}_Y, \underline{d}_Z$ and \underline{d}_M such that $\underline{d}_M = \underline{d}_X + \underline{d}_Y + \underline{d}_Z$, we have*

$$\int_{\overline{L} \in \mathbb{E}_{\underline{d}_X + \underline{d}_Y}(A)/G_{\underline{d}_X + \underline{d}_Y}} g_{XY}^L g_{LZ}^M = \int_{\overline{L'} \in \mathbb{E}_{\underline{d}_Y + \underline{d}_Z}(A)/G_{\underline{d}_Y + \underline{d}_Z}} g_{XL'}^M g_{Y'Z}^{L'}$$

Define

$$W(X, Y; L_1, L_2) :=$$

$$\{(f, g, h) \mid 0 \longrightarrow Y \xrightarrow{f} L_1 \xrightarrow{g} L_2 \xrightarrow{h} X \longrightarrow 0 \text{ is an exact sequence}\}.$$

Under the action of $G_{\underline{\alpha}} \times G_{\underline{\beta}}$, where $\underline{\alpha} = \underline{\dim} X$ and $\underline{\beta} = \underline{\dim} Y$, the orbit space is denoted by $\mathcal{V}(X, Y; L_1, L_2)$. In fact,

$$\mathcal{V}(X, Y; L_1, L_2) = \{g : L_1 \rightarrow L_2 \mid \text{Kerg} \cong Y \text{ and } \text{Cokerg} \cong X\}$$

Put

$$h_{XY}^{L_1 L_2} = \chi(\mathcal{V}(X, Y; L_1 L_2))$$

We have the following “higher order” associativity.

Theorem 4.5. *For fixed A -modules X, Y, L_i for $i = 1, 2$, we have*

$$\int_{\overline{Y}} g_{Y_2 Y_1}^Y h_{XY}^{L_1 L_2} = \int_{\overline{L'_1}} g_{L'_1 Y_1}^{L_1} h_{XY_2}^{L'_1 L_2}.$$

Dually, for fixed A -modules X_i, Y, L_i for $i = 1, 2$, we have

$$\int_{\overline{X}} g_{X_2 X_1}^X h_{XY}^{L_1 L_2} = \int_{\overline{L'_2}} g_{X_2 L'_2}^{L_2} h_{X_1 Y}^{L_1 L'_2}.$$

Proof. Define

$$\begin{aligned} EF(X, Y_1, Y_2; L_1, L_2) &= \{(g, Y^\bullet) \mid g : L_1 \rightarrow L_2, Y^\bullet = (\text{Kerg} \supseteq Y' \supseteq 0) \\ &\text{such that } \text{Cokerg} \simeq X, Y' \simeq Y_1, \text{Kerg}/Y' \simeq Y_2\} \end{aligned}$$

and

$$\begin{aligned} EF'(X, Y_1, Y_2; L'_1, L_2) &= \{(g', L'_1) \mid g' : L'_1 \rightarrow L_2, L^\bullet = (L_1 \supseteq Y' \supseteq 0) \\ &\text{such that } \text{Kerg}' \simeq Y_2, \text{Cokerg}' \simeq X, Y' \simeq Y_1, L_1/Y' \simeq L'_1\} \end{aligned}$$

Consider the following diagram

$$(12) \quad \begin{array}{ccccccccc} & & Y_1 & \xlongequal{\quad} & Y_1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \text{Ker } g & \longrightarrow & L_1 & \xrightarrow{g} & L_2 & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel & & \parallel \\ 0 & \longrightarrow & Y_2 & \longrightarrow & L'_1 & \xrightarrow{g'} & L_2 & \longrightarrow & X \longrightarrow 0 \end{array}$$

where $L'_1 = L_1/Y'$ is the pushout. This gives the following morphism of varieties:

$$EF(X, Y_1, Y_2; L_1, L_2) \rightarrow EF'(X, Y_1, Y_2; L'_1, L_2)$$

sending (g, Y^\bullet) to (g', L_1^\bullet) where $g' : L_1/Y' \rightarrow L_2$. Conversely, we also have the morphism

$$EF'(X, Y_1, Y_2; L'_1, L_2) \rightarrow EF(X, Y_1, Y_2; L_1, L_2)$$

sending (g', L_1^\bullet) to (g, Y^\bullet) where g is the composition: $L_1 \rightarrow L_1/Y' \simeq L'_1 \xrightarrow{g'} L_2$ (this implies $Y' \subseteq \text{Ker } g$). A simple check shows there exists a homeomorphism between $EF(X, Y_1, Y_2; L_1, L_2)$ and $EF'(X, Y_1, Y_2; L'_1, L_2)$. By Proposition 2.5, we have

$$\chi(EF(X, Y_1, Y_2; L_1, L_2)) = \int_Y g_{Y_2 Y_1}^Y h_{X Y}^{L_1 L_2}$$

and

$$\chi(EF'(X, Y_1, Y_2; L'_1, L_2)) = \int_{L'_1} g_{L'_1 Y_1}^{L_1} h_{X Y_2}^{L'_1 L_2}$$

This completes the proof. \square

We define

$$\text{Hom}(L_1, L_2)_{Y[1] \oplus X} = \{g \in \text{Hom}(L_1, L_2) \mid \text{Ker } g \simeq Y, \text{Coker } g \simeq X\}$$

Then, it is easy to identify that

$$\mathcal{V}(X, Y; L_1, L_2) = \text{Hom}(L_1, L_2)_{Y[1] \oplus X}$$

We can consider a \mathbb{C}^* -action on $\text{Hom}(L_1, L_2)_{Y[1] \oplus X}$ or $\mathcal{V}(X, Y; L_1, L_2)$ simply by $t.(f, g, h)^* = (f, tg, h)^*$ for $t \in \mathbb{C}^*$ and $(f, g, h)^* \in \mathcal{V}(X, Y; L_1, L_2)$. We also have a projective version of Theorem 4.5, where \mathbb{P} indicates the corresponding orbit space under the \mathbb{C}^* -action.

Theorem 4.6. *For fixed A -modules X, Y_i, L_i for $i = 1, 2$, we have*

$$\int_Y g_{Y_2 Y_1}^Y \chi(\mathbb{P}\text{Hom}(L_1, L_2)_{Y[1] \oplus X}) = \int_{L'_1} g_{L'_1 Y_1}^{L_1} \chi(\mathbb{P}\text{Hom}(L'_1, L_2)_{Y_2[1] \oplus X}).$$

Dually, for fixed A -modules X_i, Y, L_i for $i = 1, 2$, we have

$$\int_X g_{X_2 X_1}^X \chi(\mathbb{P}\text{Hom}(L_1, L_2)_{Y[1] \oplus X}) = \int_{L'_2} g_{X_2 L'_2}^{L_2} \chi(\mathbb{P}\text{Hom}(L_1, L'_2)_{Y[1] \oplus X_1}).$$

4.3. For fixed ξ, η and ξ', η' with $\underline{\xi} + \underline{\eta} = \underline{\xi}' + \underline{\eta}' = \underline{\lambda}$, let $V_\lambda \in \mathbb{E}_\lambda$ and $Q(V_\lambda)$ be the set of (a, b, a', b') such that the row and the column of the following diagram are exact:

$$(13) \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & V_\eta & & & \\ & & & \downarrow a' & & & \\ 0 & \longrightarrow & V_{\eta'} & \xrightarrow{a} & V_\lambda & \xrightarrow{b} & V_{\xi'} \longrightarrow 0 \\ & & & & \downarrow b' & & \\ & & & & V_\xi & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

We let

$$Q(\xi, \eta, \xi', \eta') = \bigcup_{V_\lambda \in \mathbb{E}_\lambda} Q(V_\lambda)$$

We remark that $Q(\xi, \eta, \xi', \eta')$ can be viewed as a constructible subset of the module variety $\mathbb{E}_{(\underline{\xi}, \underline{\eta}, \underline{\xi}', \underline{\eta}', \underline{\lambda})}$ with $\underline{\xi} + \underline{\eta} = \underline{\xi}' + \underline{\eta}' = \underline{\lambda}$ of the following quiver

$$(14) \quad \begin{array}{ccccc} & & 2 & & \\ & & \downarrow & & \\ 4 & \longrightarrow & 5 & \longrightarrow & 3 \\ & & \downarrow & & \\ & & 1 & & \end{array}$$

We have the following action of G_λ on $Q(\xi, \eta, \xi', \eta')$:

$$g.(a, b, a', b') = (ga, bg^{-1}, ga', b'g^{-1}).$$

The orbit space of $Q(\xi, \eta, \xi', \eta')$ is denoted by $Q(\xi, \eta, \xi', \eta')^*$ and the orbit of (a, b, a', b') in $Q(\xi, \eta, \xi', \eta')^*$ is denoted by $(a, b, a', b')^*$. We also have the following action of G_λ on $W(V_{\xi'}, V_{\eta'}; \mathbb{E}_\lambda)$: $g.(a, b) = (ga, bg^{-1})$. In the induced orbit space $\text{Ext}^1(V_{\xi'}, V_{\eta'})$, the orbit of (a, b) is denoted by $(a, b)^*$. Hence, we have

$$(15) \quad \begin{array}{ccc} W(V_{\xi'}, V_{\eta'}; \mathbb{E}_\lambda) \times W(\xi, \eta; \mathbb{E}_\lambda) = Q(\xi, \eta, \xi', \eta') & \xrightarrow{\phi_2} & \text{Ext}^1(V_{\xi'}, V_{\eta'}) \\ \phi_1 \downarrow & \nearrow \phi & \\ Q(\xi, \eta, \xi', \eta')^* & & \end{array}$$

where $\phi((a, b, a', b')^*) = (a, b)^*$ is well defined.

Let $(a, b, a', b') \in Q(V_\lambda)$. We claim that the stabilizer of (a, b, a', b') in ϕ_1 is

$$a'e_1 \text{Hom}(\text{Coker } b'a, \text{Ker } ba')e_4b',$$

which is isomorphic to $\text{Hom}(\text{Coker } b'a, \text{Ker } ba')$, where the injection $e_1 : \text{Ker } ba' \rightarrow V_\lambda$ is induced naturally by a' and the surjection $e_4 : V_\lambda \rightarrow \text{Coker } b'a$ is induced naturally by b' . In fact, consider the action of G_λ on $W(V_\xi, V_\eta; \mathbb{E}_\lambda)$ given by $g.(a', b') = (ga', b'g^{-1})$, the stabilizer of (a', b') is $1 + a'\text{Hom}(V_\xi, V_\eta)b'$ ([Rin2]). It is

clear that the stabilizer of (a, b, a', b') under the action given by ϕ_1 is the following subgroup:

$$\{1 + a'fb' \mid f \in \text{Hom}(V_\xi, V_\eta), ba'fb' = 0, a'fb'a = 0\}.$$

Since b' is surjective and a' is injective, $ba'fb' = 0, a'fb'a = 0$ imply $ba'f = 0, fb'a = 0$. This means $\text{Im } f \in \text{Ker } ba'$ and $f(\text{Ker } b'a) = 0$. We easily deduce the above claim by this conclusion. In the same way, the stabilizer of (a, b) under the action given by ϕ_2 is $1 + a\text{Hom}(V_{\xi'}, V_{\eta'})b$, which is isomorphic to $\text{Hom}(V_{\xi'}, V_{\eta'})$ just for a is injective and b is surjective. We now compute the fibre $\phi_1(\phi_2^{-1}((a, b)^*))$ of ϕ over $(a, b)^*$.

$$\phi_2^{-1}((a, b)^*) = (ga, bg^{-1}, a', b')$$

where $(a', b') \in W(V_\xi, V_\eta; V_\lambda)$. Fixed a, b , Fix a, b , and let $U = \{(a, b, a', b')\}$. Then

$$U \subset \phi_2^{-1}((a, b)^*), \quad \phi_1(U) = \phi_1(\phi_2^{-1}((a, b)^*))$$

$\phi_1|_U: U \rightarrow \phi_1(U)$ can be viewed as the action of the group $a\text{Hom}(V_{\xi'}, V_{\eta'})b$ with the stable subgroup $a'e_1\text{Hom}(\text{Coker } b'a, \text{Ker } ba')e_4b'$, i.e., the fibre of $\phi_1|_U$ is isomorphic to $a\text{Hom}(V_{\xi'}, V_{\eta'})b/a'e_1\text{Hom}(\text{Coker } b'a, \text{Ker } ba')e_4b'$. Hence, by Corollary 2.6,

$$\chi(W(V_\xi, V_\eta; V_\lambda)) = \chi(U) = \chi(\phi_1(U)) = \chi(\phi_1(\phi_2^{-1}((a, b)^*))).$$

Moreover, consider the action of $G_\xi \times G_\eta$ on $Q(\xi, \eta, \xi', \eta')^*$ and the induced orbit space, denoted by $Q(\xi, \eta, \xi', \eta')^\wedge$. The stabilizer $\text{Stab}_G((a, b, a', b')^*)$ of $(a, b, a', b')^*$ is

$$\{(g_1, g_2) \in G_\xi \times G_\eta \mid ga' = a'g_2, b'g = g_1b' \text{ for some } g \in 1 + a\text{Hom}(V_{\xi'}, V_{\eta'})b\},$$

also denoted by $G((a, b, a', b')^*)$. This determines the group embedding

$$\text{Stab}_G((a, b, a', b')^*) \rightarrow (1 + a\text{Hom}(V_{\xi'}, V_{\eta'})b)/(1 + ae_1\text{Hom}(\text{Coker } b'a, \text{Ker } ba')e_4b).$$

The group $G((a, b, a', b')^*)$ is isomorphic to a vector space since $ba = 0$. We know that $1 + a\text{Hom}(V_{\xi'}, V_{\eta'})b$ is the subgroup of $\text{Aut } V_\lambda$, it acts on $W(V_\xi, V_\eta; \mathbb{E}_\lambda)$ naturally. The orbit space of $W(V_\xi, V_\eta; \mathbb{E}_\lambda)$ under the action of $1 + a\text{Hom}(V_{\xi'}, V_{\eta'})b$ is denoted by $\widetilde{W}(V_\xi, V_\eta; \mathbb{E}_\lambda)$ and similar considerations hold for $\mathcal{V}(V_\xi, V_\eta; \mathbb{E}_\lambda)$. Combined with the discussion above, we have the following commutative diagram of actions of groups:

$$(16) \quad \begin{array}{ccc} W(V_\xi, V_\eta; \mathbb{E}_\lambda) & \xrightarrow{1+a\text{Hom}(V_{\xi'}, V_{\eta'})b} & \widetilde{W}(V_\xi, V_\eta; \mathbb{E}_\lambda) \\ \downarrow G_\xi \times G_\eta & & \downarrow G_\xi \times G_\eta \\ \mathcal{V}(V_\xi, V_\eta; \mathbb{E}_\lambda) & \xrightarrow{1+a\text{Hom}(V_{\xi'}, V_{\eta'})b} & \widetilde{\mathcal{V}}(V_\xi, V_\eta; \mathbb{E}_\lambda) \end{array}$$

The stabilizer of $(a', b')^\wedge$ in the bottom map is

$$\{g \in 1 + a\text{Hom}(V_{\xi'}, V_{\eta'})b \mid ga' = a'g_2, b'g = g_1b' \text{ for some } (g_1, g_2) \in G_\xi \times G_\eta\}$$

which is isomorphic to a vector space too, it is denoted by $V(a, b, a', b')$. We can construct the map from $V(a, b, a', b')$ to $\text{Stab}_G((a, b, a', b')^*)$ sending g to (g_1, g_2) . It is well-defined since a' is injective and b' is surjective. We have

$$V(a, b, a', b')/\text{Hom}(\text{Coker } b'a, \text{Ker } ba') \cong \text{Stab}_G((a, b, a', b')^*).$$

We have the following proposition.

Proposition 4.7. *The fiber over $(a, b)^* \in \text{Ext}^1(V_{\xi'}, V_{\eta'})_\lambda$ of the surjective map*

$$\phi^\wedge : Q(\xi, \eta, \xi', \eta')^\wedge \rightarrow \text{Ext}^1(V_{\xi'}, V_{\eta'})$$

is isomorphic to $\tilde{\mathcal{V}}(V_\xi, V_\eta; \mathbb{E}_\lambda)$, where $\tilde{\mathcal{V}}(V_\xi, V_\eta; \mathbb{E}_\lambda)$ is such that there exists a surjective morphism from $\mathcal{V}(V_\xi, V_\eta; V_\lambda)$ to $\tilde{\mathcal{V}}(V_\xi, V_\eta; \mathbb{E}_\lambda)$ such that any fibre is isomorphic to an affine space of dimension

$$\dim_{\mathbb{C}} \text{Hom}(V_{\xi'}, V_{\eta'}) - \dim_{\mathbb{C}} \text{Hom}(\text{Coker } b'a, \text{Ker } ba') - \dim_{\mathbb{C}} \text{Stab}_G((a, b, a', b')^*).$$

Also, we also have a commutative diagram induced by (15). By Proposition 2.5, we have

Corollary 4.8. *The following equality holds.*

$$(17) \quad \sum_{\lambda} \chi(Q(\lambda)^\wedge) = \sum_{\lambda} g_{\beta\alpha}^\lambda h_{\lambda}^{\xi'\eta'}$$

4.4. Let $\mathcal{O}(\xi, \eta, \xi', \eta')$ be the set of $(V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d)$ such that the following commutative diagram has exact rows and columns:

$$(18) \quad \begin{array}{ccccccc} & & 0 & & & 0 & \\ & & \downarrow & & & \downarrow & \\ 0 & \longrightarrow & V_\beta & \xrightarrow{e_1} & V_\eta & \xrightarrow{e_2} & V_\delta \longrightarrow 0 \\ & & \downarrow u' & \nearrow S & \downarrow u_{V_\eta} & & \downarrow x \\ & & V_{\eta'} & \xrightarrow{u_{V_{\eta'}}} & V_\lambda & \xrightarrow{\quad} & V_{\xi'} \\ & & \downarrow v' & & \downarrow c & \nearrow T & \downarrow y \\ 0 & \longrightarrow & V_\alpha & \xrightarrow{e_3} & V_\xi & \xrightarrow{e_4} & V_\gamma \longrightarrow 0 \\ & & \downarrow & & \downarrow q_{V_\xi} & \nearrow q_{V_{\xi'}} & \downarrow \\ & & 0 & & & & 0 \end{array}$$

where V_δ, V_β are submodules of $V_{\xi'}, V_{\eta'}$, respectively; $V_\gamma = V_{\xi'}/V_\delta$, $V_\alpha = V_{\eta'}/V_\beta$, u', x, v', y are the canonical morphisms, and V_λ is the center induced by the above square, $T = V_\xi \times_{V_\gamma} V_{\xi'} = \{(x \oplus m) \in V_\xi \oplus V_{\xi'} \mid e_4(x) = y(m)\}$ and $S = V_\eta \sqcup_{V_\beta} V_{\eta'} = V_\eta \oplus V_{\eta'} / \{e_1(v_\beta) \oplus u'(v_\beta) \mid v_\beta \in V_\beta\}$. Then there is unique map $f : S \rightarrow T$ for the fixed square. Let (c, d) be a pair of maps such that c is surjective, d is injective and $cd = f$. In particular, for fixed submodules V_δ and V_β of $V_{\xi'}$ and $V_{\eta'}$ respectively, the subset of $\mathcal{O}(\xi, \eta, \xi', \eta')$

$$\{(V_1, V_2, e_1, e_2, e_3, e_4, c, d) \in \mathcal{O}(\xi, \eta, \xi', \eta') \mid V_1 = V_\delta, V_2 = V_\beta\}$$

is denoted by $\mathcal{O}_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}$ where $V_\gamma = V_{\xi'}/V_\delta$ and $V_\alpha = V_{\eta'}/V_\beta$. There is a natural action of the group G_Δ on $\mathcal{O}_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}$ as follows:

$$g \cdot (V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d) = (V_\delta, V_\beta, e_1, e_2, e_3, e_4, cg^{-1}, gd)$$

We denote by $\mathcal{O}_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}^*$ and $\mathcal{O}(\xi, \eta, \xi', \eta')^*$ the orbit spaces under the actions of G_Δ .

4.5. There is a homeomorphism between $Q(\xi, \eta, \xi', \eta')^*$ and $\mathcal{O}(\xi, \eta, \xi', \eta')^*$ (see [DXX]):

$$\theta^* : Q(\xi, \eta, \xi', \eta')^* \rightarrow \mathcal{O}(\xi, \eta, \xi', \eta')^*$$

induced by the map between $Q(\xi, \eta, \xi', \eta')$ and $\mathcal{O}(\xi, \eta, \xi', \eta')$ defined as follows:

$$V_\beta = \text{Ker } b'a \simeq \text{Ker } ba', \quad V_\delta = \text{Im } ba',$$

$$e_1 = (a')^{-1}a, \quad e_2 = ba', \quad e_3 = b'a, \quad e_4 = b(b')^{-1}$$

and c, d are induced by the maps:

$$V_\eta \oplus V_{\eta'} \rightarrow V_\lambda \quad \text{and} \quad V_\lambda \rightarrow V_\xi \oplus V_{\xi'}$$

There is an action of $G_\xi \times G_\eta$ on $\mathcal{O}(\xi, \eta, \xi', \eta')^*$, defined as follows: for $(g_1, g_2) \in G_\xi \times G_\eta$,

$$(g_1, g_2) \cdot (V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d)^* = (V_\delta, V_\beta, g_2 e_1, e_2 g_2^{-1}, g_1 e_3, e_4 g_1^{-1}, c', d')^*$$

Let us determine the relation between (c', d') and (c, d) .

Suppose that $(V_\delta, V_\beta, g_2 e_1, e_2 g_2^{-1}, g_1 e_3, e_4 g_1^{-1})$ induces S', T' and the unique map $f' : S' \rightarrow T'$, then it is clear that there are isomorphisms:

$$a_1 : S \rightarrow S' \quad \text{and} \quad a_2 : T \rightarrow T'$$

induced by isomorphisms:

$$\begin{pmatrix} g_2 & 0 \\ 0 & id \end{pmatrix} : V_\eta \oplus V_{\eta'} \rightarrow V_\eta \oplus V_{\eta'} \quad \text{and} \quad \begin{pmatrix} g_1 & 0 \\ 0 & id \end{pmatrix} : V_\xi \oplus V_{\xi'} \rightarrow V_\xi \oplus V_{\xi'}$$

So $f' = a_2 f a_1^{-1}$, we have the following commutative diagram:

$$(19) \quad \begin{array}{ccccc} S' & \xrightarrow{d'} & V_\lambda & \xrightarrow{d'_1} & V_\gamma \\ \uparrow a_1 & & \uparrow g & & \parallel \\ S & \xrightarrow{d} & V_\lambda & \xrightarrow{d_1} & V_\gamma \\ \downarrow & & \downarrow c & & \parallel \\ \text{Im } f & \longrightarrow & T & \longrightarrow & V_\gamma \\ \downarrow a_2 & & \downarrow a_2 & & \parallel \\ \text{Im } f' & \longrightarrow & T' & \longrightarrow & V_\gamma \end{array}$$

Hence, $c' = a_2 c g^{-1}$ and $d' = g d a_1^{-1}$. In particular, $c = c'$ and $d = d'$ if and only if $g_1 = id_{V_\xi}$ and $g_2 = id_{V_\eta}$. This shows that the action of $G_\xi \times G_\eta$ is free.

Its orbit space is denoted by $\mathcal{O}(\xi, \eta, \xi', \eta')^\wedge$. The homeomorphism θ^* above induces the homeomorphism in the following the Proposition:

Proposition 4.9. *There exists a homeomorphism under quotient topology*

$$\theta^\wedge : Q(\xi, \eta, \xi', \eta')^\wedge \rightarrow \mathcal{O}(\xi, \eta, \xi', \eta')^\wedge.$$

Let $\mathcal{D}(\xi, \eta, \xi', \eta')^*$ be the set of $(V_\delta, V_\beta, e_1, e_2, e_3, e_4)$ such that the diagram (18) is commutative and has exact rows and columns. In particular, for fixed V_δ and V_β , its subset

$$\{(V_1, V_2, e_1, e_2, e_3, e_4) \mid V_1 = V_\delta, V_2 = V_\beta\}$$

is denoted by $\mathcal{D}_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}^*$ where $V_\gamma = V_{\xi'}/V_\delta$ and $V_\alpha = V_{\eta'}/V_\beta$. Then we have a projection:

$$\varphi^* : \mathcal{O}(\xi, \eta, \xi', \eta')^* \rightarrow \mathcal{D}(\xi, \eta, \xi', \eta')^*$$

We claim that the fibre of this morphism is isomorphic to a vector space which has the same dimension as $\text{Ext}^1(V_\gamma, V_\beta)$ for any element in $\mathcal{D}_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}^*$.

Fix an element $(V_\delta, V_\beta, e_1, e_2, e_3, e_4) \in \mathcal{D}_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}^*$, and let V be the set consisting of the equivalence classes $(c, d)^*$ of elements (c, d) under the action of G_λ such that the following diagram is commutative:

$$(20) \quad \begin{array}{ccccc} V_\beta & \xlongequal{\quad} & V_\beta & \longrightarrow & 0 \\ \downarrow u_1 & & \downarrow s & & \downarrow \\ S & \xrightarrow{d} & V_\lambda & \xrightarrow{t} & V_\gamma \\ \downarrow v_1 & & \downarrow c & & \parallel \\ L & \xrightarrow{u_2} & T & \xrightarrow{v_2} & V_\gamma \end{array}$$

where u_1, u_2, v_1, v_2 are fixed and come from the long exact sequence:

$$(21) \quad \begin{array}{ccccccc} & & L = \text{Im } f & & & & \\ & v_1 \nearrow & & \searrow u_2 & & & \\ 0 & \longrightarrow & V_\beta & \xrightarrow{u_1} & S & \xrightarrow{f} & T & \xrightarrow{v_2} & V_\gamma & \longrightarrow & 0 \end{array}$$

and where s, t are naturally induced by c, d respectively. Then V is the fibre of φ^* . We note that c is in diagram (20) if and only if $c \in u_2 \text{Hom}(V_\gamma, L)t$. Of course, $u_2 \text{Hom}(V_\gamma, L)t$ is isomorphic to $\text{Hom}(V_\gamma, L)$.

Let $\varepsilon_0 \in \text{Ext}^1(V_\gamma, L)$ be the class of the following exact sequence:

$$0 \longrightarrow L \xrightarrow{u_2} T \xrightarrow{v_2} V_\gamma \longrightarrow 0$$

The above long exact sequence induces the following long exact sequence:

$$(22) \quad \begin{aligned} 0 &\longrightarrow \text{Hom}(V_\gamma, V_\beta) \longrightarrow \text{Hom}(V_\gamma, S) \longrightarrow \text{Hom}(V_\gamma, L) \longrightarrow \\ &\longrightarrow \text{Ext}^1(V_\gamma, V_\beta) \longrightarrow \text{Ext}^1(V_\gamma, S) \xrightarrow{\phi} \text{Ext}^1(V_\gamma, L) \longrightarrow 0. \end{aligned}$$

Consider the morphism $\omega : V \rightarrow \phi^{-1}(\varepsilon_0)$ sending $(c, d)^*$ to $(d, t)^*$.

$$\omega^{-1}((d, t)^*) = \{(cg^{-1}, gd)^* \mid g \in G_\lambda\} = \{(cg^{-1}, d)^* \mid g \in 1 + d\text{Hom}(V_\gamma, S)t\}$$

Hence, the fibre of ω can be viewed as the orbit space of $u_2 \text{Hom}(V_\gamma, L)t$ under the action of $1 + d\text{Hom}(V_\gamma, S)t$ given by $g \cdot u_2 f t = u_2 f g_\gamma^{-1} t$ where $g \in 1 + d\text{Hom}(V_\gamma, S)t$ and g_γ is the isomorphism on V_γ induced by g . with the stabilizer isomorphic to the vector space $\text{Hom}(V_\gamma, V_\beta)$. Hence, up to a translation from ε_0 to 0, V is isomorphic to the affine space

$$\phi^{-1}(\varepsilon_0) \times \text{Hom}(V_\gamma, L) \times \text{Hom}(V_\gamma, V_\beta) / \text{Hom}(V_\gamma, S)$$

which is denoted by $W(V_\delta, V_\beta, e_1, e_2, e_3, e_4)$, and whose dimension is $\dim_{\mathbb{C}} \text{Ext}^1(V_\gamma, V_\beta)$.

There is also an action of the group $G_\xi \times G_\eta$ on $\mathcal{D}(\xi, \eta, \xi', \eta')^*$ with stabilizer isomorphic to the vector space $\text{Hom}(V_\gamma, V_\alpha) \times \text{Hom}(V_\delta, V_\beta)$. The orbit space is denoted by $\mathcal{D}(\xi, \eta, \xi', \eta')^\wedge$. The projection φ^* naturally induces the projection:

$$\varphi^\wedge : \mathcal{O}(\xi, \eta, \xi', \eta')^\wedge \rightarrow \mathcal{D}(\xi, \eta, \xi', \eta')^\wedge$$

Its fibre over $(V_\delta, V_\beta, e_1, e_2, e_3, e_4)^\wedge$ is isomorphic to the quotient space of

$$(\varphi^*)^{-1}(V_\delta, V_\beta, e_1, e_2, e_3, e_4)$$

under the action of $\text{Hom}(V_\gamma, V_\alpha) \times \text{Hom}(V_\delta, V_\beta)$. The corresponding stabilizer of $(V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d)^* \in (\varphi^*)^{-1}(V_\delta, V_\beta, e_1, e_2, e_3, e_4)$ is

$$\begin{aligned} &\{(g_1, g_2) \in 1 + e_3 \text{Hom}(V_\gamma, V_\alpha) e_4 \times 1 + e_1 \text{Hom}(V_\delta, V_\beta) e_2 \mid \\ &ga' = a'g_2, b'g = g_1b' \text{ for some } g \in 1 + a\text{Hom}(V_{\xi'}, V_{\eta'})b\} \end{aligned}$$

where $(a, b, a', b')^*$ is induced by θ^* as showed in diagram (18). It is isomorphic to the vector space $\text{Stab}_G((a, b, a', b')^*)$. Therefore, we have

Proposition 4.10. *There exists a projection*

$$\varphi^\wedge : \mathcal{O}(\xi, \eta, \xi', \eta')^\wedge \rightarrow \mathcal{D}(\xi, \eta, \xi', \eta')^\wedge$$

such that any fibre for $(V_\delta, V_\beta, e_1, e_2, e_3, e_4)^\wedge$ is isomorphic to an affine space of dimension

$$\dim_{\mathbb{C}} \text{Ext}^1(V_\gamma, V_\beta) + \dim_{\mathbb{C}} \text{Stab}_G((a, b, a', b')^*) - \dim_{\mathbb{C}} \text{Hom}(V_\gamma, V_\alpha) - \dim_{\mathbb{C}} \text{Hom}(V_\delta, V_\beta)$$

where $V_\gamma \simeq V_{\xi'}/V_\delta$ and $V_\alpha \simeq V_{\eta'}/V_\beta$.

Let us summarize the discussion above in the following diagram:

(23)

$$\text{Ext}^1(V_{\xi'}, V_{\eta'}) \xleftarrow{\phi^\wedge} Q(\xi, \eta, \xi', \eta')^\wedge \xrightarrow{\theta^\wedge} \mathcal{O}(\xi, \eta, \xi', \eta')^\wedge \xrightarrow{\varphi^\wedge} \mathcal{D}(\xi, \eta, \xi', \eta')^\wedge.$$

The following theorem can be viewed as a degenerated version of Green's formula.

Theorem 4.11. *For fixed ξ, η, ξ', η' , we have*

$$g_{\xi\eta}^{\xi' \oplus \eta'} = \int_{\alpha, \beta, \delta, \gamma; \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'}$$

Proof. We note that

$$\chi(\mathcal{D}(\xi, \eta, \xi', \eta')^\wedge) = \int_{\alpha, \beta, \delta, \gamma} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} h_\xi^{\gamma\alpha} h_\eta^{\delta\beta}$$

Because the Euler characteristic of an affine space is 1, we have

$$\int_{\lambda} h_\lambda^{\xi' \eta'} g_{\xi\eta}^\lambda = \int_{\alpha, \beta, \delta, \gamma} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} h_\xi^{\gamma\alpha} h_\eta^{\delta\beta}$$

Using Proposition 2.5 and Lemma 4.2, we simplify the identity as

$$h_{\xi' \oplus \eta'}^{\xi' \eta'} g_{\xi\eta}^{\xi' \oplus \eta'} = \int_{\alpha, \beta, \delta, \gamma; \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} h_\xi^{\gamma\alpha} h_\eta^{\delta\beta}$$

i.e.,

$$g_{\xi\eta}^{\xi' \oplus \eta'} = \int_{\beta, \delta} g_{\xi'/\delta, \delta}^{\xi'} g_{\eta'/\beta, \beta}^{\eta'}.$$

□

4.6. We define $EF(\xi, \eta, \xi', \eta')$ to be the set

$$\{(\varepsilon, L'(d)) \mid \varepsilon \in \text{Ext}^1(V_{\xi'}, V_{\eta'})_{L(d)}, L'(d) \subseteq L(d), L'(d) \simeq V_\eta, L(d)/L'(d) \simeq V_\xi\}$$

and let $FE(\xi, \eta, \xi', \eta')$ be the set

$$\{(V_{\xi'}, V_{\eta'}, \varepsilon_1, \varepsilon_2) \mid V_{\xi'} \subseteq V_\xi, V_{\eta'} \subseteq V_{\eta'}, \varepsilon_1 \in \text{Ext}^1(V_{\xi'}, V_{\eta'})_{V_\eta}, \varepsilon_2 \in \text{Ext}^1(V_{\xi'}/V_{\xi'}, V_{\eta'}/V_{\eta'})_{V_\xi}\}.$$

The projection

$$p_1 : EF(\xi, \eta, \xi', \eta') \rightarrow \text{Ext}^1(V_{\xi'}, V_{\eta'})$$

satisfies that the fibre of any $\varepsilon \in \text{Ext}^1(V_{\xi'}, V_{\eta'})_{L(d)}$ is isomorphic to $\mathcal{V}(V_\xi, V_\eta; L(d))$.

Comparing with Proposition 4.7, we have a morphism

$$EF(\xi, \eta, \xi', \eta') \rightarrow Q(\xi, \eta, \xi', \eta')^\wedge$$

satisfying the fibre of $(a, b, a', b')^\wedge$ is isomorphic to an affine space of dimension

$$\dim_{\mathbb{C}} \text{Hom}(V_{\xi'}, V_{\eta'}) - \dim_{\mathbb{C}} \text{Hom}(\text{Coker } b'a, \text{Ker } b'a') - \dim_{\mathbb{C}} G((a, b, a', b')^*).$$

We also have a natural homeomorphism

$$FE(\xi, \eta, \xi', \eta') \rightarrow \mathcal{D}(\xi, \eta, \xi', \eta')^\wedge.$$

Hence, using Proposition 4.7, 4.9 and 4.10, we have

Proposition 4.12. *There is a natural morphism*

$$\rho : EF(\xi, \eta, \xi', \eta') \rightarrow FE(\xi, \eta, \xi', \eta')$$

satisfying the fibre for $(V_{\xi'}, V_{\eta'}, \varepsilon_1, \varepsilon_2)$ is isomorphic to an affine space of dimension

$$\dim_{\mathbb{C}} \text{Hom}(V_{\xi'}, V_{\eta'}) - \dim_{\mathbb{C}} \text{Hom}(V_{\gamma}, V_{\beta}) + \dim_{\mathbb{C}} \text{Ext}(V_{\gamma}, V_{\beta}) - \dim_{\mathbb{C}} \text{Hom}(V_{\gamma}, V_{\alpha}) - \dim_{\mathbb{C}} \text{Hom}(V_{\delta}, V_{\beta}).$$

where $V_{\beta} \simeq V_{\eta'}, V_{\delta} \simeq V_{\xi'}$ and $V_{\alpha} \simeq V_{\eta'}/V_{\eta'}, V_{\gamma} \simeq V_{\xi'}/V_{\xi'}$.

Now we consider the action of \mathbb{C}^* on $EF(\xi, \eta, \xi', \eta')$ and $FE(\xi, \eta, \xi', \eta')$.

(1) For $t \in \mathbb{C}^*$ and $(\varepsilon, L'(d)) \in EF(\xi, \eta, \xi', \eta')$, we know $t.\varepsilon \in \text{Ext}^1(V_{\xi'}, V_{\eta'})_{L(t.d)}$ and $L(d) = V_{\xi'} \oplus V_{\eta'}$ as a direct sum of vector spaces. Recall that $L(t.d)$ is defined in Section 4.1. Define

$$L'(t.d) := \{(v', tv'') \mid (v', v'') \in L(d)\}.$$

Then $L'(t.d) \subseteq L(t.d)$. Hence, we define

$$t.(\varepsilon, L'(d)) = (t.\varepsilon, L'(t.d)).$$

The orbit space is denoted by $\widehat{EF}(\xi, \eta, \xi', \eta')$. A point $(\varepsilon, L'(d))$ is stable, i.e., $t.(\varepsilon, L'(d)) = (\varepsilon, L'(d))$ for any $t \in \mathbb{C}^*$ if and only if

$$L(d) = V_{\xi'} \oplus V_{\eta'} \text{ and } L'(d) = (L'(d) \cap V_{\xi'}) \oplus (L'(d) \cap V_{\eta'})$$

Note that the above direct sums are the direct sums of modules. The set of stable points in $EF(\xi, \eta, \xi', \eta')$ is denoted by $EF_s(\xi, \eta, \xi', \eta')$. The action of \mathbb{C}^* on the set of non-stable points in $EF(\xi, \eta, \xi', \eta')$ is free. We denote the orbit space by $\mathbb{P}EF(\xi, \eta, \xi', \eta')$. Of course, we have

$$\widehat{EF}(\xi, \eta, \xi', \eta') = EF_s(\xi, \eta, \xi', \eta') \bigcup \mathbb{P}EF(\xi, \eta, \xi', \eta').$$

The orbit of $(\varepsilon, L'(d))$ in $\mathbb{P}EF(\xi, \eta, \xi', \eta')$ is denoted by $\mathbb{P}(\varepsilon, L'(d))$.

(2) For $t \in \mathbb{C}^*$ and $(V_{\xi'}, V_{\eta'}, \varepsilon_1, \varepsilon_2) \in FE(\xi, \eta, \xi', \eta')$, we define

$$t.(V_{\xi'}, V_{\eta'}, \varepsilon_1, \varepsilon_2) = (V_{\xi'}, V_{\eta'}, t.\varepsilon_1, t.\varepsilon_2).$$

The orbit space is denoted by $\widehat{FE}(\xi, \eta, \xi', \eta')$. A point $(V_{\xi'}, V_{\eta'}, \varepsilon_1, \varepsilon_2)$ in $FE(\xi, \eta, \xi', \eta')$ is stable if and only if $\varepsilon_1 = \varepsilon_2 = 0$. The set of stable points in $FE(\xi, \eta, \xi', \eta')$ is denoted by $FE_s(\xi, \eta, \xi', \eta')$. The action of \mathbb{C}^* on the set of non-stable points in $FE(\xi, \eta, \xi', \eta')$ is free. We denote the orbit space by $\mathbb{P}FE(\xi, \eta, \xi', \eta')$. Of course, we have

$$\widehat{FE}(\xi, \eta, \xi', \eta') = FE_s(\xi, \eta, \xi', \eta') \bigcup \mathbb{P}FE(\xi, \eta, \xi', \eta').$$

The orbit of $(V_{\xi'}, V_{\eta'}, \varepsilon_1, \varepsilon_2)$ in $\mathbb{P}FE(\xi, \eta, \xi', \eta')$ is denoted by $\mathbb{P}(V_{\xi'}, V_{\eta'}, \varepsilon_1, \varepsilon_2)$.

The morphism ρ induces the morphism

$$\hat{\rho} : \widehat{EF}(\xi, \eta, \xi', \eta') \rightarrow \widehat{FE}(\xi, \eta, \xi', \eta')$$

We consider its restriction to $\mathbb{P}EF(\xi, \eta, \xi', \eta')$.

$$\hat{\rho} \big|_{\mathbb{P}EF(\xi, \eta, \xi', \eta')} : \mathbb{P}EF(\xi, \eta, \xi', \eta') \rightarrow \widehat{FE}(\xi, \eta, \xi', \eta')$$

For any $(V_{\xi'}, V_{\eta'}, 0, 0) \in FE_s(\xi, \eta, \xi', \eta') \subseteq \widehat{FE}(\xi, \eta, \xi', \eta')$, we have

$$(\hat{\rho} \big|_{\mathbb{P}EF(\xi, \eta, \xi', \eta')})^{-1}((V_{\xi'}, V_{\eta'}, 0, 0)) = \mathbb{P}(\rho^{-1}(V_{\xi'}, V_{\eta'}, 0, 0) \setminus (V_{\xi'} \oplus V_{\eta'}, 0)).$$

It is actually the projective space of the affine space in Proposition 4.12. For any $\mathbb{P}(V_{\xi'}, V_{\eta'}, \varepsilon_1, \varepsilon_2) \in \mathbb{P}FE(\xi, \eta, \xi', \eta')$, $(\hat{\rho} \big|_{\mathbb{P}EF(\xi, \eta, \xi', \eta')})^{-1}(\mathbb{P}(V_{\xi'}, V_{\eta'}, \varepsilon_1, \varepsilon_2))$ is isomorphic to the affine space in Proposition 4.12. Now we compute the Euler characteristics. By Proposition 2.1 and the above discussion of the fibres, we have

$$\chi(\mathbb{P}EF(\xi, \eta, \xi', \eta')) =$$

$$\int_{(V'_{\xi'}, V'_{\eta'}, 0, 0) \in FE_s(\xi, \eta, \xi', \eta')} [d(\xi', \eta') - d(\gamma, \alpha) - d(\delta, \beta) - \langle \gamma, \beta \rangle] g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} + \chi(\mathbb{P}FE(\xi, \eta, \xi', \eta')).$$

where $d(\gamma, \alpha) = \dim_{\mathbb{C}} \text{Hom}_{\Lambda}(V_{\gamma}, V_{\alpha})$ and the Euler form $\langle \gamma, \beta \rangle = \dim_{\mathbb{C}} \text{Hom}(V_{\gamma}, V_{\beta}) - \dim_{\mathbb{C}} \text{Ext}^1(V_{\gamma}, V_{\beta})$. On the other hand, we know

$$\begin{aligned} \chi(\mathbb{P}EF(\xi, \eta, \xi', \eta')) = \\ \int_{\alpha, \beta, \delta, \gamma, \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} \chi(\mathbb{P}\overline{V}(V_{\xi}, V_{\eta}; V_{\xi'} \oplus V_{\eta'})) + \int_{\lambda \neq \xi' \oplus \eta'} \chi(\mathbb{P}Ext^1(V_{\xi'}, V_{\eta'})_{\lambda}) g_{\xi\eta}^{\lambda}. \end{aligned}$$

Therefore, we have the following theorem, which can be viewed as a geometric version of Green's formula under the \mathbb{C}^* -action.

Theorem 4.13. *For fixed ξ, η, ξ', η' , we have*

$$\begin{aligned} & \int_{\lambda \neq \xi' \oplus \eta'} \chi(\mathbb{P}Ext^1(V_{\xi'}, V_{\eta'})_{\lambda}) g_{\xi\eta}^{\lambda} = \\ & \int_{\alpha, \beta, \delta, \gamma, \alpha \oplus \gamma \neq \xi \text{ or } \beta \oplus \delta \neq \eta} \chi(\mathbb{P}(\text{Ext}^1(V_{\gamma}, V_{\alpha})_{\xi} \times \text{Ext}^1(V_{\delta}, V_{\beta})_{\eta})) g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} \\ & + \int_{\alpha, \beta, \delta, \gamma, \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} [d(\xi', \eta') - d(\gamma, \alpha) - d(\delta, \beta) - \langle \gamma, \beta \rangle] g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} \\ & - \int_{\alpha, \beta, \delta, \gamma, \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} \chi(\mathbb{P}\overline{V}(V_{\xi}, V_{\eta}; V_{\xi'} \oplus V_{\eta'})). \end{aligned}$$

5. APPLICATION TO CALDERO-KELLER FORMULA

5.1. Let Q be a quiver with vertex set $Q_0 = \{1, 2, \dots, n\}$ containing no oriented cycles and $A = \mathbb{C}Q$ be the path algebra of Q . For $i \in Q_0$, we denote by P_i the corresponding indecomposable projective $\mathbb{C}Q$ -module and by S_i the corresponding simple module. Let $\mathbb{Q}(x_1, \dots, x_n)$ be a transcendental extension of \mathbb{Q} . Define the map

$$X_{\tau} : \text{obj}(\text{mod}A) \rightarrow \mathbb{Q}(x_1, \dots, x_n)$$

by:

$$X_M = \sum_{\underline{e}} \chi(Gr_{\underline{e}}(M)) x^{\tau(\underline{e}) - \underline{\dim} M + \underline{e}}$$

where τ is the Auslander-Reiten translation on the Grothendieck group $K_0(\mathcal{D}^b(Q))$ and, for $v \in \mathbb{Z}^n$, we put

$$x^v = \prod_{i=1}^n x_i^{\langle \underline{\dim} S_i, v \rangle}$$

and $Gr_{\underline{e}}(M)$ is the \underline{e} -Grassmannian of M , i.e. the variety of submodules of M with dimension vector \underline{e} . This definition is equivalent to [Hu2]

$$X_M = \int_{\alpha, \beta} g_{\alpha\beta}^M x^{\beta R + \alpha R' - \underline{\dim} M}$$

where the matrices $R = (r_{ij})$ and $R' = (r'_{ij})$ satisfy $r_{ij} = \dim_{\mathbb{C}} \text{Ext}^1(S_i, S_j)$ and $r'_{ij} = \dim_{\mathbb{C}} \text{Ext}^1(S_j, S_i)$ for $i, j \in Q_0$. Here we recall $g_{\alpha\beta}^M := \chi(\mathcal{V}((V_{\alpha}, V_{\beta}; M)))$ which is defined in Section 4.1. Note that (see [Hu2])

$$(\underline{\dim} P)R = \underline{\dim} \text{rad } P \quad (\underline{\dim} I)R' = \underline{\dim} I - \underline{\dim} \text{soc } I$$

We consider the set

$$Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}) := \{(M, M_1) \mid M \in \mathbb{E}_{\underline{d}}, M_1 \in Gr_{\underline{e}}(M)\}.$$

This is a closed subset of $\mathbb{E}_{\underline{d}} \times \prod_{i=1, \dots, n} Gr_{e_i}(k^{d_i})$. Here, we simply use the notation $\mathbb{E}_{\underline{d}}$ instead of $\mathbb{E}_{\underline{d}}(Q)$ without confusion.

Proposition 5.1. *The function $X_\gamma|_{\mathbb{E}_{\underline{d}}}$ is G -invariant constructible.*

Proof. Obviously it is G -invariant. Consider the canonical morphism $\pi : Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}) \rightarrow \mathbb{E}_{\underline{d}}$ sending (M, M_1) to M . It is clear that $\pi^{-1}(M) = Gr_{\underline{e}}(M)$. Let $1_{Gr_{\underline{e}}(\mathbb{E}_{\underline{d}})}$ be the constant function on $Gr_{\underline{e}}(\mathbb{E}_{\underline{d}})$, by Theorem 2.2, $(\pi)_*(1_{Gr_{\underline{e}}(\mathbb{E}_{\underline{d}})})$ is constructible. We know that

$$(\pi)_*(1_{Gr_{\underline{e}}(\mathbb{E}_{\underline{d}})})(M) = \chi(Gr_{\underline{e}}(M))$$

So there are finitely many $\chi(Gr_{\underline{e}}(M))$ for $M \in \mathbb{E}_{\underline{d}}$. \square

Proposition 5.2. *For fixed dimension vectors \underline{e} and \underline{d} , the set*

$$\{g_{XY}^E \mid E \in \mathbb{E}_{\underline{d}}, Y \in \mathbb{E}_{\underline{e}}, X \in \mathbb{E}_{\underline{d}-\underline{e}}\}$$

is a finite set.

Proof. Let $M \in \mathbb{E}_{\underline{d}}$. For any submodule M_1 of dimension vector \underline{e} of M , by the knowledge of linear algebra, there exist unique $(\mathbb{C}^{\underline{e}}, x) \in \mathbb{E}_{\underline{e}}$ isomorphic to M_1 and $(\mathbb{C}^{\underline{d}-\underline{e}}, x') \in \mathbb{E}_{\underline{d}-\underline{e}}$ isomorphic to M/M_1 , this deduces the following morphisms:

$$Gr_{\underline{e}}(M) \xrightarrow{\pi_1} \mathbb{E}_{\underline{e}} \times \mathbb{E}_{\underline{d}-\underline{e}} \times \mathbb{E}_{\underline{d}} \xrightarrow{\pi_2} \bigcup_i \phi_i(U_i)$$

where $\mathbb{E}_{\underline{e}} \times \mathbb{E}_{\underline{d}-\underline{e}} \times \mathbb{E}_{\underline{d}} = \bigcup_i U_i$ is a finite stratification with respect to the action of the algebraic group $G_{\underline{e}} \times G_{\underline{d}-\underline{e}}$ and $\phi_i : U_i \rightarrow \phi_i(U_i)$ is the geometric quotient for any i , and $\pi_2 = \bigcup_i \phi_i$. For any $(Y, X, M) \in \mathbb{E}_{\underline{e}} \times \mathbb{E}_{\underline{d}-\underline{e}} \times \mathbb{E}_{\underline{d}}$,

$$\chi((\pi_2 \pi_1)^{-1}(\pi_2((Y, X, M)))) = g_{XY}^M.$$

Consider the constant function $1_{Gr_{\underline{e}}(M)}$ on $Gr_{\underline{e}}(M)$, by Theorem 2.2, $(\pi_2 \pi_1)_*(1_{Gr_{\underline{e}}(M)})$ is constructible. Hence, there are finitely many g_{XY}^M for $(X, Y, M) \in \mathbb{E}_{\underline{e}} \times \mathbb{E}_{\underline{d}-\underline{e}} \times \mathbb{E}_{\underline{d}}$. \square

Proposition 5.3. *For fixed $M \in \mathbb{E}_{\underline{\xi}'}$, $N \in \mathbb{E}_{\underline{\eta}'}$, the set*

$$\{\chi(\text{Ext}^1(M, N)_E) \mid E \in \mathbb{E}_{\underline{\xi}'+\underline{\eta}'}\}$$

is a finite set.

Proof. Consider the morphism:

$$\text{Ext}^1(M, N) \xrightarrow{f} \mathbb{E}_{\underline{\xi}'+\underline{\eta}'} \xrightarrow{g} \bigcup_j \phi_j(V_j)$$

where $\mathbb{E}_{\underline{\xi}'+\underline{\eta}'} = \bigcup_j V_j$ is a finite stratification with respect to the action of the algebraic group $G_{\underline{\xi}'+\underline{\eta}'}$ and $\phi_j : V_j \rightarrow \phi_j(V_j)$ is a geometric quotient for any j , and f sends any extension to the middle term of the extension. Here, f is a morphism by identification between $\text{Ext}^1(M, N)$ and $E(M, N)$ at the beginning of Section 4. The remaining discussion is almost the same as in Proposition 5.2. We omit it. \square

5.2. We now consider the cluster category, i.e. the orbit category $\mathcal{D}^b(Q)/F$ with $F = [1]\tau^{-1}$, where τ is the AR-translation of $\mathcal{D}^b(Q)$. Each object M in $\mathcal{D}^b(Q)/F$ can be uniquely decomposed into the form: $M = M_0 \oplus P_M[1] = M_0 \oplus \tau P_M$ where $M_0 \in \text{mod } A$ and P_M is projective in $\text{mod } A$. Now we can extend the map X_γ as in [CK], see also [Hu2] by the rule: $X_{\tau P} = x_{\underline{\dim} P / \text{rad } P}$ for projective A -module and $X_{M \oplus N} = X_M X_N$. Then we have a well-defined map

$$X_\gamma : \text{obj}(\mathcal{D}^b(Q)/F) \rightarrow \mathbb{Q}(x_1, \dots, x_n).$$

Let $\overline{\mathbb{E}}_{\underline{d}}$ be the orbit space of $\mathbb{E}_{\underline{d}}$ under the action of $G_{\underline{d}}$. Note that all the integrals below are over $\overline{\mathbb{E}}_{\underline{d}}$ for some corresponding dimension vector \underline{d} . Note also that in $\text{mod } A$ we have $d(\gamma, \alpha) = \dim \text{Hom}(V_\gamma, V_\alpha)$ and $d^1(\gamma, \alpha) = \dim \text{Ext}(V_\gamma, V_\alpha)$. We

say that P_0 is the projective direct summand of $V_{\xi'}$ if $V_{\xi'} \simeq V'_{\xi'} \oplus P_0$ and no direct summand of $V'_{\xi'}$ is projective.

The cluster algebra corresponding to the cluster category $\mathcal{D}^b(Q)/F$ is the subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by $\{X_M, X_{\tau P} | M \in \text{mod } A, P \in \text{mod } A \text{ is projective}\}$. The following theorem gives a generalization of the cluster multiplication formula in [CK]. The idea of the proof follows the work [Hu2] of Hubery.

Theorem 5.4. (1) For any A -modules $V_{\xi'}$, $V_{\eta'}$ we have

$$\begin{aligned} d^1(\xi', \eta') X_{V_{\xi'}} X_{V_{\eta'}} &= \int_{\lambda \neq \xi' \oplus \eta'} \chi(\mathbb{P}\text{Ext}^1(V_{\xi'}, V_{\eta'})_{V_\lambda}) X_{V_\lambda} \\ &+ \int_{\gamma, \beta, \iota} \chi(\mathbb{P}\text{Hom}(V_{\eta'}, \tau V_{\xi'})_{V_\beta[1] \oplus \tau V'_\gamma \oplus I_0}) X_{V_\gamma} X_{V_\beta} x^{\underline{\dim} \text{soc } I_0} \end{aligned}$$

where $I_0 \in \iota$ is injective and $V_\gamma = V'_\gamma \oplus P_0$, P_0 is the projective direct summand of $V_{\xi'}$.

(2) For any A -module $V_{\xi'}$ and $P \in \rho$ is projective Then

$$\begin{aligned} d(\rho, \xi') X_{V_{\xi'}} x^{\underline{\dim} P / \text{rad } P} &= \int_{\delta, \iota'} \chi(\mathbb{P}\text{Hom}(V_{\xi'}, I)_{V_\delta[1] \oplus I'}) X_{V_\delta} x^{\underline{\dim} \text{soc } I'} \\ &+ \int_{\gamma, \rho'} \chi(\mathbb{P}\text{Hom}(P, V_{\xi'})_{P'[1] \oplus V_\gamma}) X_{V_\gamma} x^{\underline{\dim} P' / \text{rad } P'} \end{aligned}$$

where $I = D\text{Hom}(P, A)$, and $I' \in \iota'$ injective, $P' \in \rho'$ projective.

Proof. We set

$$S_1 := \int_{\lambda \in \underline{\mathbb{E}}_{\xi' + \eta'}, \lambda \neq \xi' \oplus \eta'} \chi(\mathbb{P}\text{Ext}^1(V_{\xi'}, V_{\eta'})_{V_\lambda}) X_{V_\lambda}$$

By Proposition 5.2,

$$S_1 = \int_{\xi, \eta, \lambda \neq \xi' \oplus \eta'} \chi(\mathbb{P}\text{Ext}^1(V_{\xi'}, V_{\eta'})_{V_\lambda}) g_{\xi\eta}^\lambda x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')}$$

Using Theorem 4.13, we have

$$\begin{aligned} &\int_{\xi, \eta, \lambda \neq \xi' \oplus \eta'} \chi(\mathbb{P}\text{Ext}^1(V_{\xi'}, V_{\eta'})_{V_\lambda}) g_{\xi\eta}^\lambda x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} = \\ &\int_{\alpha, \beta, \delta, \gamma, \xi, \eta, \alpha \oplus \gamma \neq \xi \text{ or } \beta \oplus \delta \neq \eta} \chi(\mathbb{P}(\text{Ext}^1(V_\gamma, V_\alpha)_{V_\xi} \times \text{Ext}^1(V_\delta, V_\beta)_{V_\eta})) g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} \\ &+ \int_{\alpha, \beta, \delta, \gamma, \xi, \eta, \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} [d(\xi', \eta') - d(\gamma, \alpha) - d(\delta, \beta) - \langle \gamma, \beta \rangle] g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} \\ &- \int_{\alpha, \beta, \delta, \gamma, \xi, \eta, \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} \chi(\mathbb{P}\overline{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})) x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} \end{aligned}$$

We come to simplify every term following [Hu2]. For fixed $\alpha, \beta, \delta, \gamma$,

$$\begin{aligned} &\int_{\xi, \eta, \alpha \oplus \gamma \neq \xi \text{ or } \beta \oplus \delta \neq \eta} \chi(\mathbb{P}(\text{Ext}^1(V_\gamma, V_\alpha)_{V_\xi} \times \text{Ext}^1(V_\delta, V_\beta)_{V_\eta})) = d^1(\gamma, \alpha) + d^1(\delta, \beta) \\ &\int_{\alpha, \beta, \delta, \gamma, \xi, \eta, \alpha \oplus \gamma \neq \xi \text{ or } \beta \oplus \delta \neq \eta} \chi(\mathbb{P}(\text{Ext}^1(V_\gamma, V_\alpha)_{V_\xi} \times \text{Ext}^1(V_\delta, V_\beta)_{V_\eta})) g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} \\ &= \int_{\alpha, \beta, \delta, \gamma} [d^1(\gamma, \alpha) + d^1(\delta, \beta)] g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} \end{aligned}$$

Moreover,

$$d^1(\gamma, \alpha) + d^1(\delta, \beta) + d(\xi', \eta') - d(\gamma, \alpha) - d(\delta, \beta) - \langle \gamma, \beta \rangle = d^1(\xi', \eta') + \langle \delta, \alpha \rangle$$

Hence,

$$\begin{aligned} & \int_{\xi, \eta, \lambda \neq \xi' \oplus \eta'} \chi(\mathbb{P}\text{Ext}^1(V_{\xi'}, V_{\eta'})_{V_\lambda}) g_{\xi\eta}^\lambda x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} = \\ & + \int_{\alpha, \beta, \delta, \gamma} [d^1(\xi', \eta') + \langle \delta, \alpha \rangle] g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} \\ & - \int_{\alpha, \beta, \delta, \gamma, \xi, \eta, \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} \chi(\mathbb{P}\overline{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})) x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} \end{aligned}$$

As for the last term, consider the following diagram, it may be compared with diagram (10)).

$$(24) \quad \bigcup_{\alpha, \beta, \gamma, \delta} \mathcal{V}(V_\alpha, V_\beta; V_{\eta'}) \times \mathcal{V}(V_\gamma, V_\delta; V_{\xi'}) \xrightarrow{j_1} \bigcup_{\xi, \eta} \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})$$

sending $(V_{\eta'}^1 \subseteq V_{\eta'}, V_{\xi'}^1 \subseteq V_{\xi'})$ to $(V_{\xi'}^1 \oplus V_{\eta'}^1 \subseteq V_{\xi'} \oplus V_{\eta'})$. And

$$(25) \quad \bigcup_{\xi, \eta} \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \xrightarrow{j_2} \bigcup_{\alpha, \beta, \gamma, \delta} \mathcal{V}(V_\alpha, V_\beta; V_{\eta'}) \times \mathcal{V}(V_\gamma, V_\delta; V_{\xi'})$$

sending $(V^1 \subseteq V_{\xi'} \oplus V_{\eta'})$ to $(V^1 \cap V_{\eta'} \subseteq V_{\eta'}, V^1/V^1 \cap V_{\eta'} \subseteq V_{\xi'})$. The map j_1 is an embedding and

$$\bigcup_{\xi, \eta} \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \setminus \text{Im } j_1 = \bigcup_{\xi, \eta} \overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})$$

The fibre of j_2 is isomorphic to a vector space $V(\delta, \alpha)$ of dimension $d(\delta, \alpha)$ (see [Hu2, Corollary 8]). If we restrict j_2 to $\bigcup_{\xi, \eta} \overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})$, then the fibre is isomorphic to $V(\delta, \alpha) \setminus \{0\}$. Under the action of \mathbb{C}^* , by Proposition 2.1, we have

$$\begin{aligned} & \int_{\alpha, \beta, \delta, \gamma, \xi, \eta, \alpha \oplus \gamma = \xi, \beta \oplus \delta = \eta} \chi(\mathbb{P}\overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})) x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} \\ & = \int_{\alpha, \beta, \delta, \gamma} d(\delta, \alpha) g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} . \end{aligned}$$

Therefore,

$$S_1 = \int_{\alpha, \beta, \delta, \gamma} [d^1(\xi', \eta') - d^1(\delta, \alpha)] g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')} .$$

There is a natural \mathbb{C}^* -action on $\text{Hom}(V_{\eta'}, \tau V_{\xi'})_{V_\beta[1] \oplus \tau V_\gamma' \oplus I_0} \setminus \{0\}$ by left multiplication, the orbit space is $\mathbb{P}\text{Hom}(V_{\eta'}, \tau V_{\xi'})_{V_\beta[1] \oplus \tau V_\gamma' \oplus I_0}$. Define

$$S_2 := \int_{\gamma, \beta, \iota, \kappa, \iota, \mu, \theta} \chi(\mathbb{P}\text{Hom}(V_{\eta'}, \tau V_{\xi'})_{V_\beta[1] \oplus \tau V_\gamma' \oplus I_0}) g_{\kappa\iota}^\gamma g_{\theta\mu}^\beta x^{(\underline{\iota} + \underline{\mu})R + (\underline{\kappa} + \underline{\theta})R' - (\underline{\beta} + \underline{\gamma}) + \underline{\text{dim}}\text{soc } I_0}$$

where $I_0 \in \iota$. The above definition is well-defined by Proposition 5.2 and 5.3. We note that

$$\underline{\text{dim}}\text{soc } I_0 - (\underline{\beta} + \underline{\gamma}) = (\underline{\xi}' - \underline{\gamma})R + (\underline{\eta}' - \underline{\beta})R' - (\underline{\xi}' + \underline{\eta}').$$

Since $\text{Hom}(V_{\eta'}, \tau V_{\xi'})_{V_{\beta}[1] \oplus \tau V_{\gamma}' \oplus I_0} = \mathcal{V}(\tau V_{\gamma}' \oplus I_0, V_{\beta}; V_{\eta'}, \tau V_{\xi'})$, we can apply Theorem 4.6 to the following diagram twice:

$$(26) \quad \begin{array}{ccccccccc} & & V_{\mu} & \xlongequal{\quad} & V_{\mu} & & \tau V_{\kappa} & \xlongequal{\quad} & \tau V_{\kappa} \\ & & \downarrow & & \downarrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & V_{\beta} & \longrightarrow & V_{\eta'} & \longrightarrow & \tau V_{\xi'} & \longrightarrow & \tau V_{\gamma}' \oplus I_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel & & \parallel \\ 0 & \longrightarrow & X & \longrightarrow & V_{\tilde{\beta}} & \longrightarrow & \tau V_{\xi'} & \longrightarrow & \tau V_{\gamma}' \oplus I_0 \longrightarrow 0 \\ & & \parallel & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & X & \longrightarrow & V_{\tilde{\beta}} & \longrightarrow & \tau V_{\gamma}' & \longrightarrow & \tau L \oplus I_0 \longrightarrow 0 \end{array}$$

where $V_{\tilde{\beta}}$ and $\tau V_{\gamma}'$ are the corresponding pullback and pushout. Moreover,

$$(\underline{\xi}' - \underline{\gamma} + \underline{l} + \underline{\mu})R + (\underline{\eta}' - \underline{\beta} + \underline{\kappa} + \underline{\theta})R' - (\underline{\xi}' + \underline{\eta}') = (\underline{\gamma} + \underline{\mu})R + (\underline{\kappa} + \underline{\beta})R' - (\underline{\xi}' + \underline{\eta}')$$

Hence,

$$\begin{aligned} S_2 &= \int_{\tilde{\gamma}, \tilde{\beta}, \kappa, \mu, l, \theta} \chi(\mathbb{P}\text{Hom}(V_{\tilde{\beta}}, \tau V_{\gamma}')_{X[1] \oplus \tau L \oplus I_0}) g_{\kappa \tilde{\gamma}}^{\xi'} g_{\tilde{\beta} \mu}^{\eta'} x^{(\underline{\gamma} + \underline{\mu})R + (\underline{\kappa} + \underline{\beta})R' - (\underline{\xi}' + \underline{\eta}')} \\ &= \int_{\tilde{\gamma}, \tilde{\beta}, \kappa, \mu} \chi(\mathbb{P}\text{Hom}(V_{\tilde{\beta}}, \tau V_{\gamma}')) g_{\kappa \tilde{\gamma}}^{\xi'} g_{\tilde{\beta} \mu}^{\eta'} x^{(\underline{\gamma} + \underline{\mu})R + (\underline{\kappa} + \underline{\beta})R' - (\underline{\xi}' + \underline{\eta}')} \\ &= \int_{\alpha, \beta, \delta, \gamma} d^1(\delta, \alpha) g_{\gamma \delta}^{\xi'} g_{\alpha \beta}^{\eta'} x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')}. \end{aligned}$$

Hence,

$$S_1 + S_2 = d^1(\xi', \eta') \int_{\alpha, \beta, \delta, \gamma} g_{\gamma \delta}^{\xi'} g_{\alpha \beta}^{\eta'} x^{\underline{\eta}R + \underline{\xi}R' - (\underline{\xi}' + \underline{\eta}')}$$

The first assertion is proved. In order to prove the second part, by Theorem 4.6, we have

$$\begin{aligned} & \int_{\delta, \delta_1, \delta_2, \nu'} g_{\delta_1 \delta_2}^{\delta} \chi(\mathbb{P}\text{Hom}(V_{\xi'}, I)_{V_{\delta}[1] \oplus I'}) x^{\underline{\delta}_2 R + \underline{\delta}_1 R' - \underline{\delta} + \underline{\dim} \text{soc } I'} \\ &= \int_{\tilde{\xi}', \delta_1, \delta_2, \nu'} g_{\tilde{\xi}' \delta_2}^{\xi'} \chi(\mathbb{P}\text{Hom}(V_{\tilde{\xi}'}, I)_{V_{\delta_1}[1] \oplus I'}) x^{\underline{\delta}_2 R + \underline{\tilde{\xi}'} R' - \underline{\xi}' + \underline{\dim} \text{soc } I} \\ &= \int_{\tilde{\xi}', \delta_2} g_{\tilde{\xi}' \delta_2}^{\xi'} \chi(\mathbb{P}\text{Hom}(V_{\tilde{\xi}'}, I)) x^{\underline{\delta}_2 R + \underline{\tilde{\xi}'} R' - \underline{\xi}' + \underline{\dim} \text{soc } I} \end{aligned}$$

and

$$\begin{aligned} & \int_{\gamma, \gamma_1, \gamma_2, \rho'} g_{\gamma_1 \gamma_2}^{\gamma} \chi(\mathbb{P}\text{Hom}(P, V_{\xi'})_{P'[1] \oplus V_{\gamma}}) x^{\underline{\gamma}_2 R + \underline{\gamma}_1 R' - \underline{\gamma} + \underline{\dim} P' / \text{rad } P'} \\ &= \int_{\tilde{\xi}', \gamma_1, \gamma_2, \rho'} g_{\gamma_1 \tilde{\xi}'}^{\xi'} \chi(\mathbb{P}\text{Hom}(P, V_{\tilde{\xi}'})_{P'[1] \oplus V_{\gamma_2}}) x^{\underline{\gamma}_1 R + \underline{\tilde{\xi}'} R' - \underline{\xi}' + \underline{\dim} P / \text{rad } P} \\ &= \int_{\tilde{\xi}', \gamma_1} g_{\tilde{\xi}' \gamma_1}^{\xi'} \chi(\mathbb{P}\text{Hom}(P, V_{\tilde{\xi}'})) x^{\underline{\gamma}_1 R + \underline{\tilde{\xi}'} R' - \underline{\xi}' + \underline{\dim} P / \text{rad } P}. \end{aligned}$$

We note that

$$\underline{\dim} \text{soc } I = \underline{\dim} P / \text{rad } P$$

and

$$\chi(\mathbb{P}\text{Hom}(P, M)) = \chi(\mathbb{P}\text{Hom}(P, V_{\tilde{\xi}'})) + \chi(\mathbb{P}\text{Hom}(V_{\tilde{\xi}'}, I)).$$

The second assertion is proved. \square

5.3. We illustrate Theorem 5.4 by the following example.

Let Q be the Kronecker quiver $1 \rightrightarrows 2$. Let S_1 and S_2 be the simple modules associated to vertices 1 and 2, respectively. Hence,

$$R = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R' = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

and

$$X_{S_1} = x^{\underline{\dim} S_1 R' - \underline{\dim} S_1} + x^{\underline{\dim} S_1 R - \underline{\dim} S_1} = x_1^{-1}(1 + x_2^2),$$

$$X_{S_2} = x^{\underline{\dim} S_2 R' - \underline{\dim} S_2} + x^{\underline{\dim} S_2 R - \underline{\dim} S_2} = x_2^{-1}(1 + x_1^2).$$

For $\lambda \in \mathbb{P}^1(\mathbb{C})$, let u_λ be the regular representation $\mathbb{C} \xrightarrow[\lambda]{1} \mathbb{C}$. Then

$$X_{u_\lambda} = x^{(1,1)R' - (1,1)} + x^{(1,1)R - (1,1)} + x^{(0,1)R + (1,0)R' - (1,1)} = x_1 x_2^{-1} + x_1^{-1} x_2 + x_1^{-1} x_2^{-1}.$$

Let I_1 and I_2 be the indecomposable injective modules corresponding vertices 1 and 2, respectively, then

$$X_{(I_1 \oplus I_2)[-1]} := x^{\underline{\dim} \text{soc}(I_1 \oplus I_2)} = x_1 x_2$$

The left side of the identity of Theorem 5.4 is

$$\dim_{\mathbb{C}} \text{Ext}^1(S_1, S_2) X_{S_1} X_{S_2} = 2(x_1^{-1} x_2^{-1} + x_1 x_2^{-1} + x_1^{-1} x_2 + x_1 x_2).$$

The first term of the right side is

$$\int_{\lambda \in \mathbb{P}^1(\mathbb{C})} \chi(\mathbb{P}\text{Ext}^1(S_1, S_2)_{u_\lambda}) X_{u_\lambda} = 2(x_1^{-1} x_2^{-1} + x_1 x_2^{-1} + x_1^{-1} x_2).$$

To compute the second term of the right side, we note that for any $f \neq 0 \in \text{Hom}(S_2, \tau S_1)$, we have the following exact sequence:

$$0 \rightarrow S_2 \xrightarrow{f} \tau S_1 \rightarrow I_1 \oplus I_2 \rightarrow 0.$$

This implies $\text{Hom}(S_2, \tau S_1) \setminus \{0\} = \text{Hom}(S_2, \tau S_1)_{I_1 \oplus I_2}$. Hence, the second term is equal to $2x_1 x_2$.

REFERENCES

- [BMRRT] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*. Advances in Math. 204 (2006), 572-618.
- [CK] P. Caldero and B. Keller, *From triangulated categories to cluster algebras*, math.RT/0506018, to appear in invent.math.
- [Di] A. Dimca, *Sheaves in topology*. Universitext. Springer-Verlag, Berlin, 2004.
- [DXX] M. Ding, J. Xiao and F. Xu, *Realizing Enveloping Algebras via Varieties of Modules*, math.QA/0604560.
- [FZ] S.Fomin, A.Zelevinsky, *Cluster algebras. I. Foundations*. J. Amer. Math. Soc. 15 (2002), no. 2, 497-529.
- [GLS] C.Geiss, B.Leclerc and J.Schrör, *Semicanonical bases and preprojective algebras II: a multiplication formula*, preprint, to appear in Compositio Math.
- [Gre] J.A. Green, *Hall algebras, hereditary algebras and quantum groups*, Invent. Math. **120**, 361-377 (1995).
- [Hu1] A. Hubery, *From triangulated categories to Lie algebras: A theorem of Peng and Xiao*, Proceedings of the Workshop on Representation Theory of Algebras and related Topics (QuerZZtaro, 2004), editors J. De la Peña and R. Bautista.
- [Hu2] A. Hubery, *Acyclic cluster algebras via Ringel-Hall algebras*, preprint.
- [Hu3] A. Hubery, *Hall Polynomials for Affine Quivers*, preprint.
- [Joy] D. Joyce, *Constructible functions on Artin stacks*, J. London Math. Soc. **74**, 583-606 (2006).
- [Lu] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. 1991, 4(2):365-421.
- [Mac] R. MacPherson, *Chern classes for singular algebraic varieties*, Ann. Math. **100**, 423-432 (1974).

- [Rie] Ch. Riedtmann, *Lie algebras generated by indecomposables*, J. Algebra **170**, 526-546(1994).
- [Rin1] C. Ringel, *Hall algebras and quantum groups*, Invent. Math.**101**, 583-592 (1990).
- [Rin2] C. Ringel, *Green's theorem on Hall algebras*, Representation theory of algebras and related topics (Mexico City, 1994), 185–245, CMS Conf. Proc., **19**, Amer. Math. Soc., Providence, RI, (1996).
- [Ro] M. Rosenlicht, *A Remark on quotient spaces*, An. Acad. Brasil. Ciênc. **35**, 487-489 (1963).
- [To] B. Toën, *Derived Hall algebras*, Duke Math. J. 135 (2006), no. 3, 587-615.
- [XX] J. Xiao, and F. Xu, *Hall algebras associated to triangulated categories*, to appear in Duke Math. Journal.
- [XXZ] J. Xiao, F. Xu and G. Zhang, *Derived categories and Lie algebras*, math.QA/0604564.

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